What is the target of a 2-functor $\mathrm{Tang}(G, X) \to$ ______ ?

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$\mathcal{C}$ ribbon category

Tang category of oriented framed tangles

$\mathrm{Tang}_{\mathcal{C}}$ tangles colored by objects of $\mathcal{C}$

Reshetikhin-Turaev Construction

There is a functor
$$\mathrm{Tang}_{\mathcal{C}} \to \mathcal{C}$$
giving tangle invariants

$\mathcal{C}$ ribbon category

G group

$\mathcal{C}$ G-ribbon category

Tang category of oriented framed tangles

Tang(G) category of oriented framed tangles with reps $\pi_1 \to G$

G-structures

$\text{Tang}_{\mathcal{C}}$ tangles coloured by objects of $\mathcal{C}$

$\text{Tang}_{\mathcal{C}}(G)$ tangles coloured by objects of $\mathcal{C}$ respecting degrees

Reshetikhin-Turaev Construction

There is a functor

$$\text{Tang}_{\mathcal{C}} \to \mathcal{C}$$

giving invariants of tangles

G-Reshetikhin-Turaev [Turaev, others?]

There is a functor

$$\text{Tang}_{\mathcal{C}}(G) \to \mathcal{C}$$

giving invariants of tangles with G-structures

Today: $\mathrm{Tang}(G,X)$ where X is a G-module

- Now a 2-category, not just monoidal (but 0-deg part is pointed, in some sense)

- Two constructions of functors from $\mathrm{Tang}(G,X)$
  1. quantization of $SL_2(\mathbb{C})$ Chern-Simons
  2. zesting and RT invariants

Question: What is the right definition of target category for these?

Background

monoidal category

product $\otimes$ on objects

$X, Y \mapsto X \otimes Y$

$f : X \to X'$, $g : Y \to Y'$

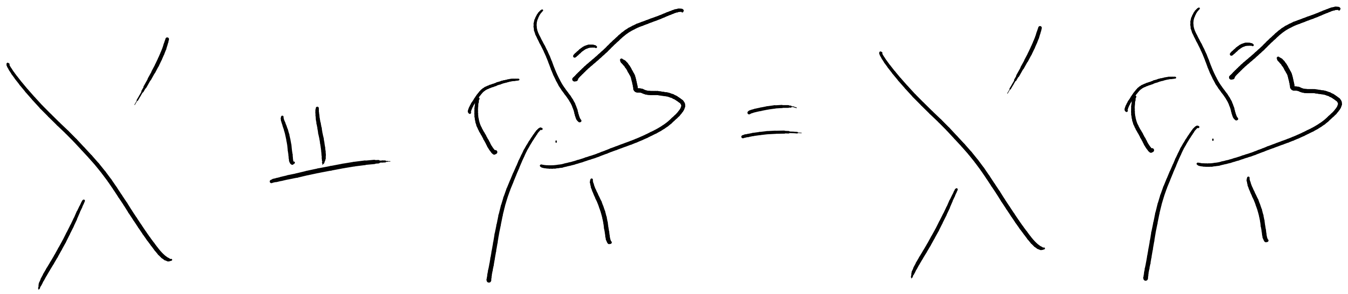
$f \otimes g : X \otimes Y \to X' \otimes Y'$

Ex vector spaces over a field
$\otimes$ is tensor product

Ex A-modules for A a Hopf alg.
product $\otimes$ comes from coproduct $\Delta : A \to A \otimes A$

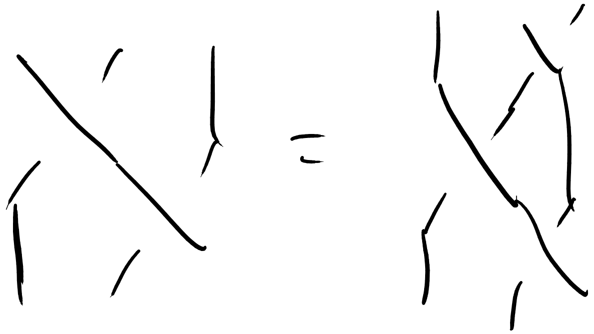
Ex cobordisms, $\otimes$ is disjoint union

Ex Tang = oriented framed tangles. Product is disjoint union!

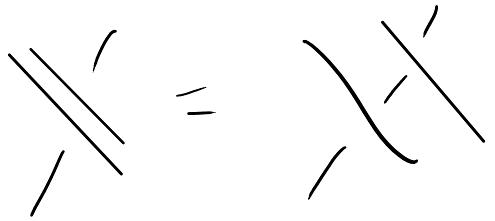


Convention: read tangles $\downarrow$ $\rightarrow$

Tang is also braided



Yang-Baxter



(one of the) hexagon rels

A braiding on a monoidal category has maps

$$\beta_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying compatibility w/ $\otimes$, "hexagon axioms" which imply braid relations

Reshetikhin–Turaev: Can interpret tangles colored by objects of a ribbon (braided + twist) category $\mathcal{C}$ as morphisms in $\mathcal{C}$.

Gives invariants of tangles.

A G-graded monoidal category (over a field k) has

objects $X \cong \bigoplus_{g \in G} X_g$, $|X_g| = g$

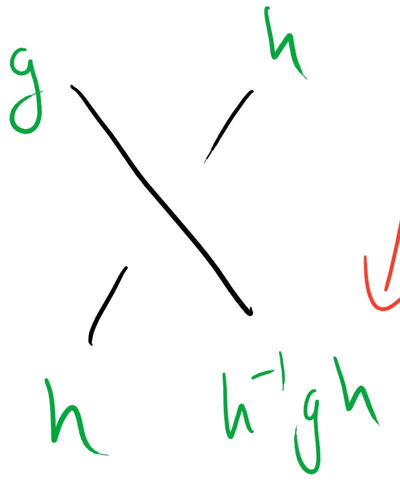
morphisms

$\mathrm{Hom}(X,Y) = 0$ unless $|X| = |Y|$

for homogeneous objects

$|X \otimes Y| = |X| \cdot |Y|$

Ex Tang(G) has tangles with reps of π_1 into G: (use Wirtinger presentation)



this is the wrong convention on π_1: I blame quantum topologists

A G-crossed braiding has

a G-action $Y \mapsto Y^g$

$|Y| = h$, $|Y^g| = hgh^{-1}$

braiding maps

$\beta : X_g \otimes Y_h \to (Y_h)^g \otimes X_g$

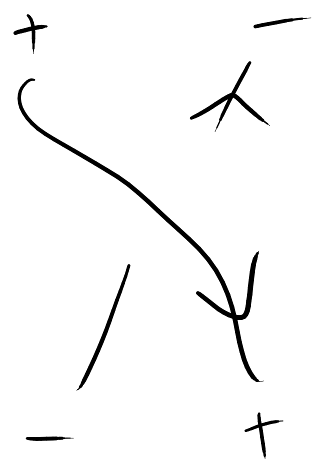
$|(Y_h)^g| = hgh^{-1}$

satisfying septagon axioms

(hexagon w/ G-actions)

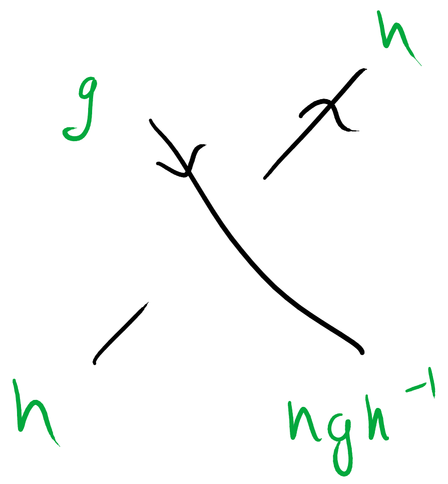
More precisely:

Tang has objects generated by $+, -$



: $(+,-) \to (-,+)$

Tang(G) has objects generated by $(g, \pm)$ for $g \in G$



: $(g,+;h,-) \to (h,-;hgh^{-1},+)$

Observation A monoidal category $(\mathcal{C}, \otimes)$ is a 2-category* with one object:

object: $*$

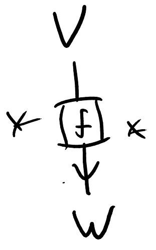
1-morphisms: objects V of $\mathcal{C}$ 1-morphism composition $\otimes$

$V : * \to *$

2-morphisms morphisms $f : V \to W$ of $\mathcal{C}$ 2-morphism composition $\circ$

$$\begin{tikzcd} * \arrow[r, "V"] \arrow[d] \arrow[dr, phantom, "\Downarrow f"] & * \arrow[d] \\ * \arrow[r, "W"'] & * \end{tikzcd}$$

dualize



Graphical calculus

$* \rightsquigarrow$ regions   dim 2

$\mathrm{Ob}(\mathcal{C}) \rightsquigarrow$ points/strings   dim 1

$\mathrm{Hom}(\mathcal{C}) \rightsquigarrow$ morphisms   dim 0

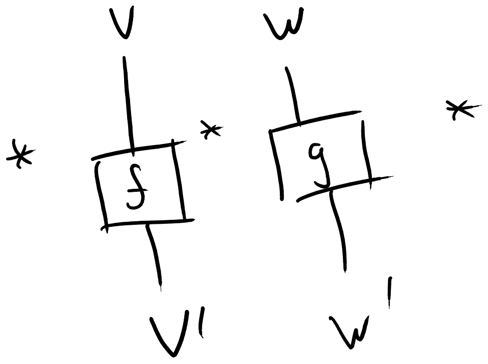
* I am going to assume strictness and probably say something wrong anyway

Now $(f \otimes g) \circ (f' \otimes g') =$

$(f \circ f') \otimes (g \circ g')$

is something about compatibility

$f \otimes g$ becomes



Since there is only one region label we usually drop it

What if we didn't drop the region labels?

Def G group, X a G-module

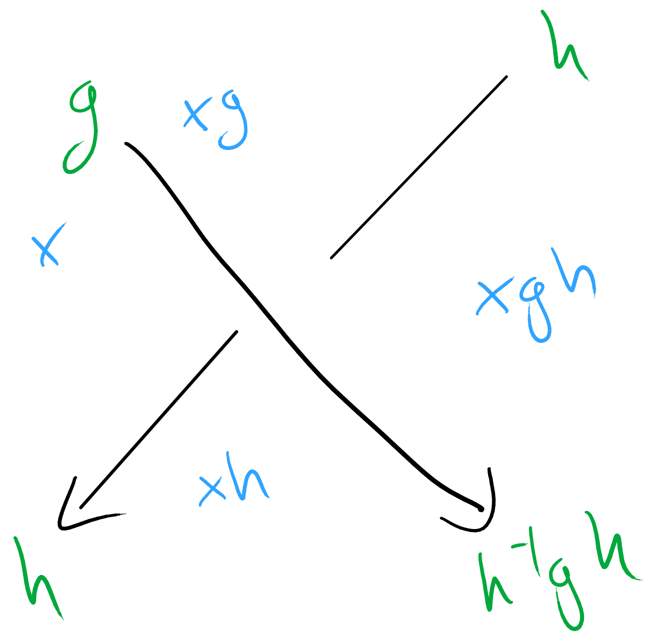
$\mathrm{Tang}(G,X)$ has

objects $x \in X$

1-morphisms For each $g \in G$, $x \in X$

$g : x \to xg$

2-morphisms Tangles plus G and X labeling



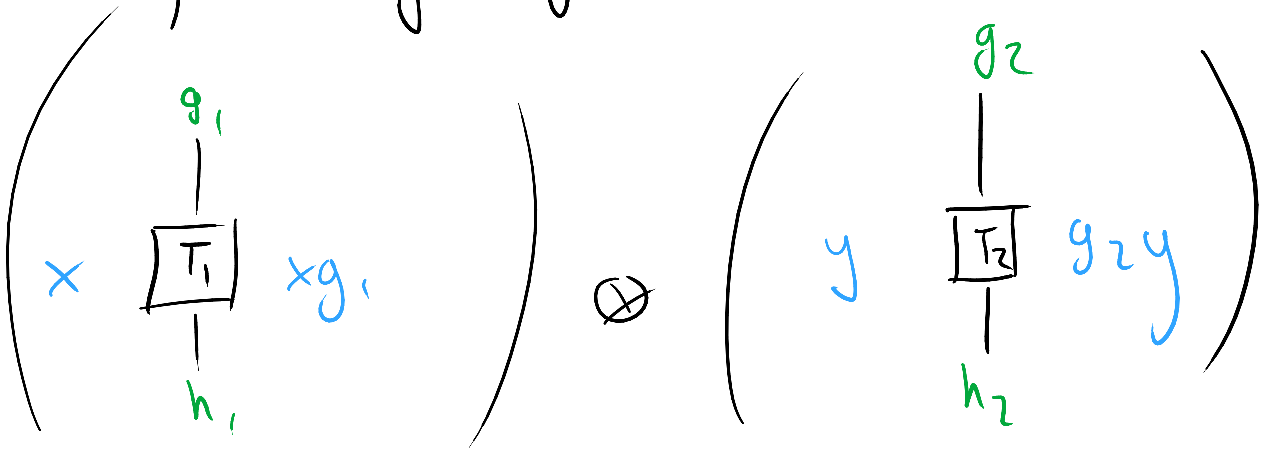
Note: region labels are determined by any one value.

Topologically a 2-morphism of $\mathrm{Tang}(G,X)$ is:

- a tangle T
- rep $\pi_1(\text{complement } T) \to G$
- gauge data $x \in X$ (of left-hand region)

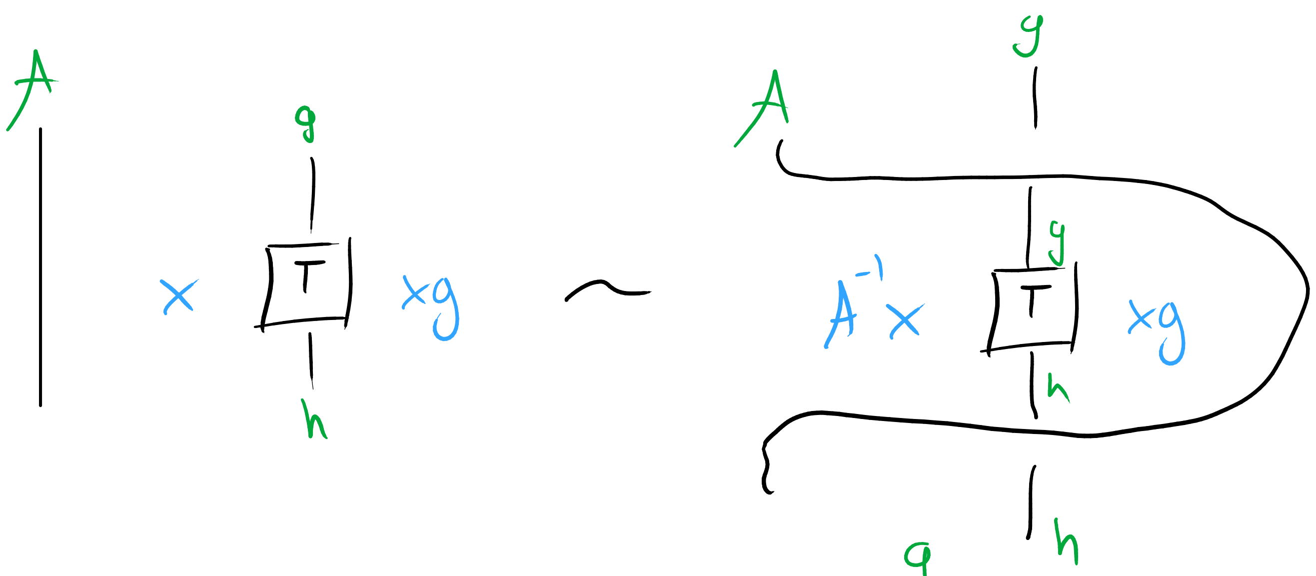
Note:

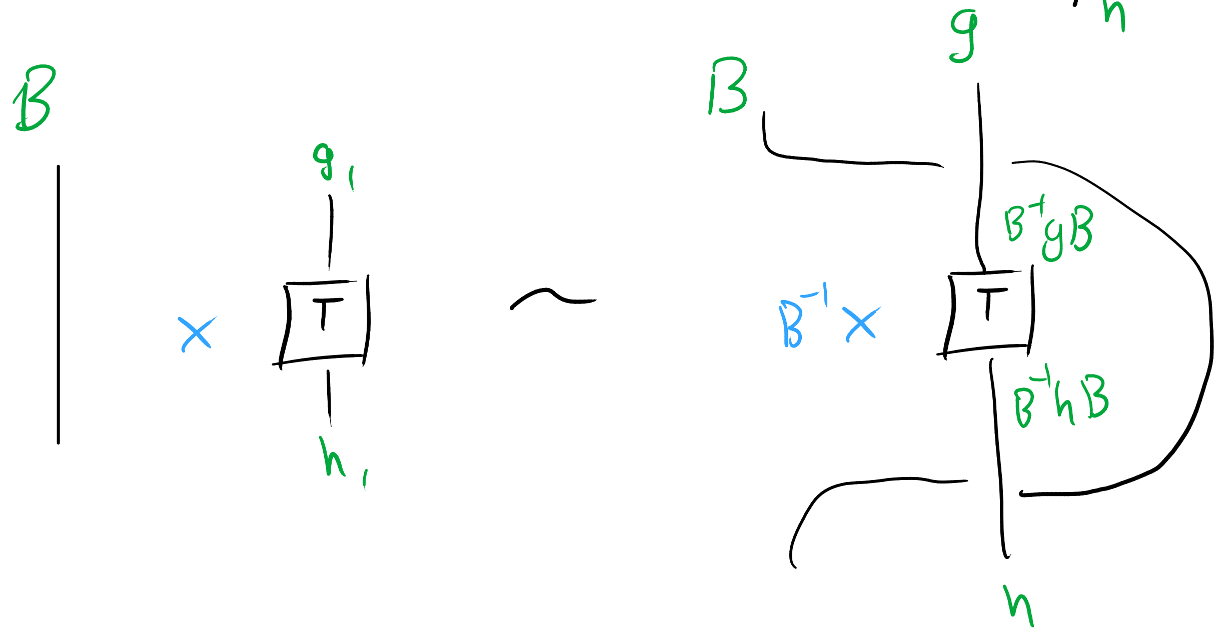
Can think of this as $\otimes$ being partially defined



is defined iff $xg_1 = y$

Braiding gives gauge transformations:





Reason 1 you should care: $SL_2(\mathbb{C})$ Chern-Simons invariants of tangles

M closed 3-mfld

$\rho : \pi_1(M) \to SL_2(\mathbb{C})$ (hyperbolic structure)

A flat sl_2 connection with holonomy ρ

$$S(A) := \frac{-i}{4}\int \mathrm{tr}[A, dA] + \frac{2}{3}\mathrm{tr}[A, A\wedge A]$$

$SL_2(\mathbb{C})$ Chern-Simons invariant

indep of A mod $4\pi^2 i$

$\mathrm{Re}(S(M,\rho)) = \mathrm{Vol}(M,\rho)$

hyperbolic volume

How to compute?

1. triangulate M
2. use ρ to find geometric shapes of tetrahedra ← nontrivial!
3. add up dilogarithms on each

(nontrivial boundary conditions here)

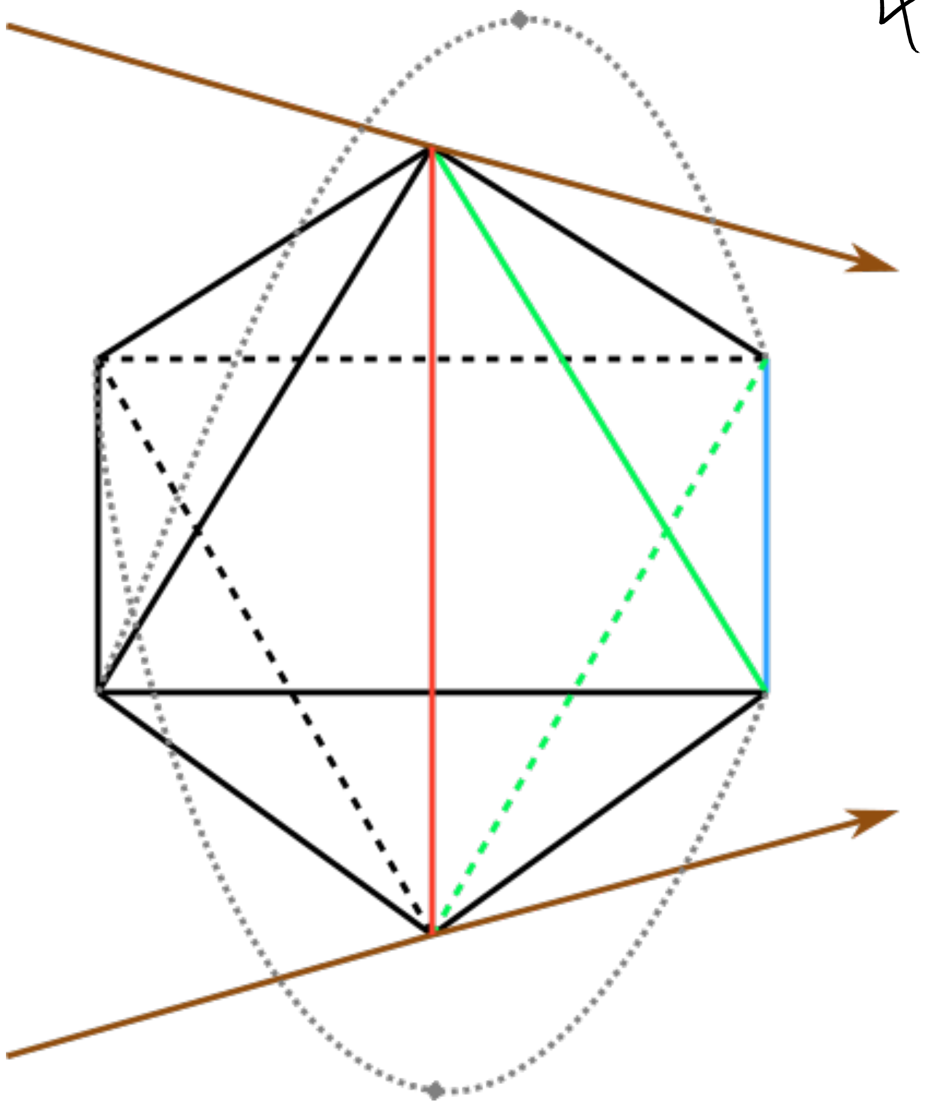
What about knots and tangles?

If M (say $S^3 \setminus \nu(K)$ knot exterior) has boundary, S depends on boundary conditions,

Now $S(M,\rho)$ takes value in line bundle over rep. var of ∂M.

How to do this for links/tangles from a diagram D?

octahedral decomposition $\mathcal{T}_D$



4 tetrahedra per crossing of D

To encode ρ in $\mathcal{T}_D$

local description of π_1 using face-pairing maps

to go back and forth need $\mathrm{Tang}(SL_2(\mathbb{C}), \mathbb{C}^2)$ to keep track of gauge data

To encode ρ in D

global description of π_1 using Wirtinger generators

Thm [Me, extending Inoue and Kabaya]

Can define 2-functor*

$$\mathcal{I} : \mathrm{Tang}(SL_2(\mathbb{C}), \mathbb{C}^2)^* \to \mathrm{Vect}$$

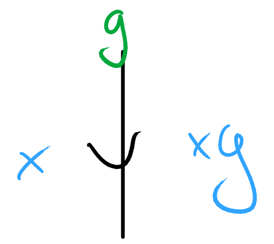
capturing $e^{\frac{CS_{\mathbb{C}}}{2\pi}}$, exponentiated $SL_2(\mathbb{C})$ Chern-Simons invariant

Need shadow coloring by elts of $\mathbb{C}^2$ to keep track of octahedral parameters

* only generically defined

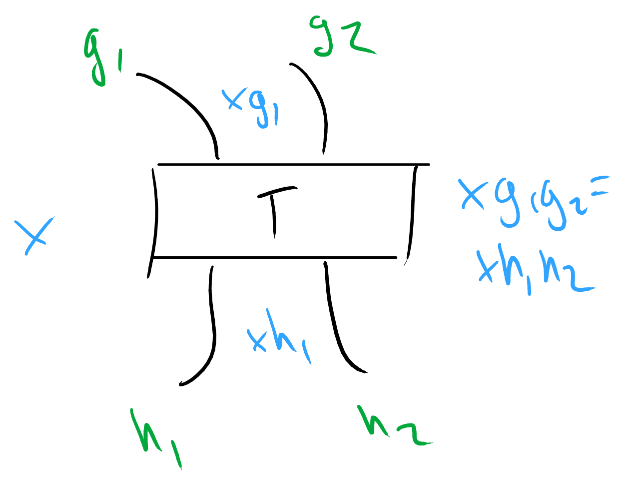
* replace $SL_2(\mathbb{C})$ by decorated quandle: pairs (g, L) with L eigenspace of g

$$x \in \mathbb{C}^2 \overset{\mathcal{I}}{\longmapsto} \mathbb{C}$$



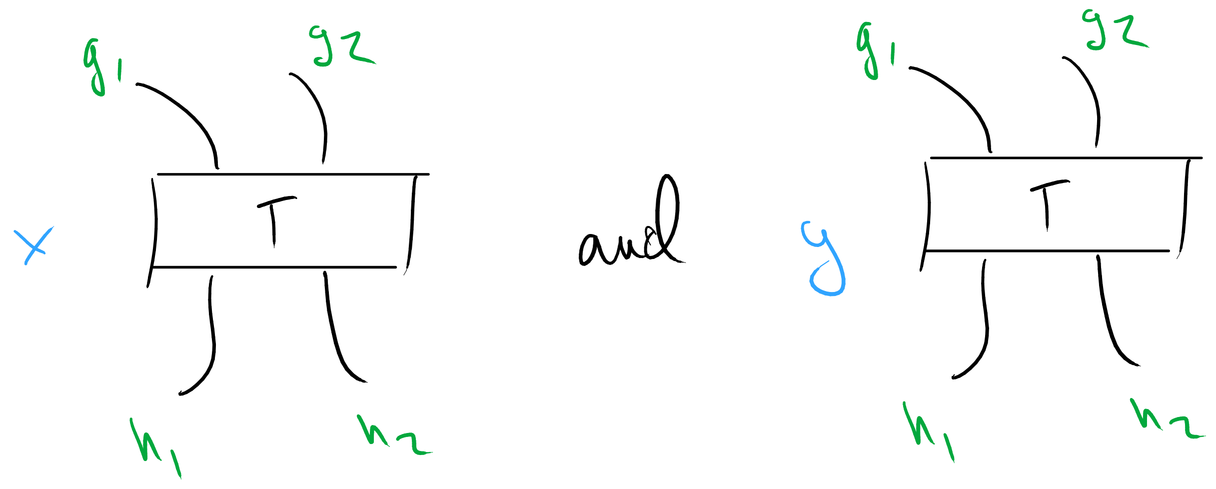
$\overset{\mathcal{I}}{\longmapsto}$ a complex line depending on x and g (not canonically $\mathbb{C}$)

so $\mathcal{I}$ is invertible



$\overset{\mathcal{I}}{\longmapsto}$ an element of product of relevant complex lines

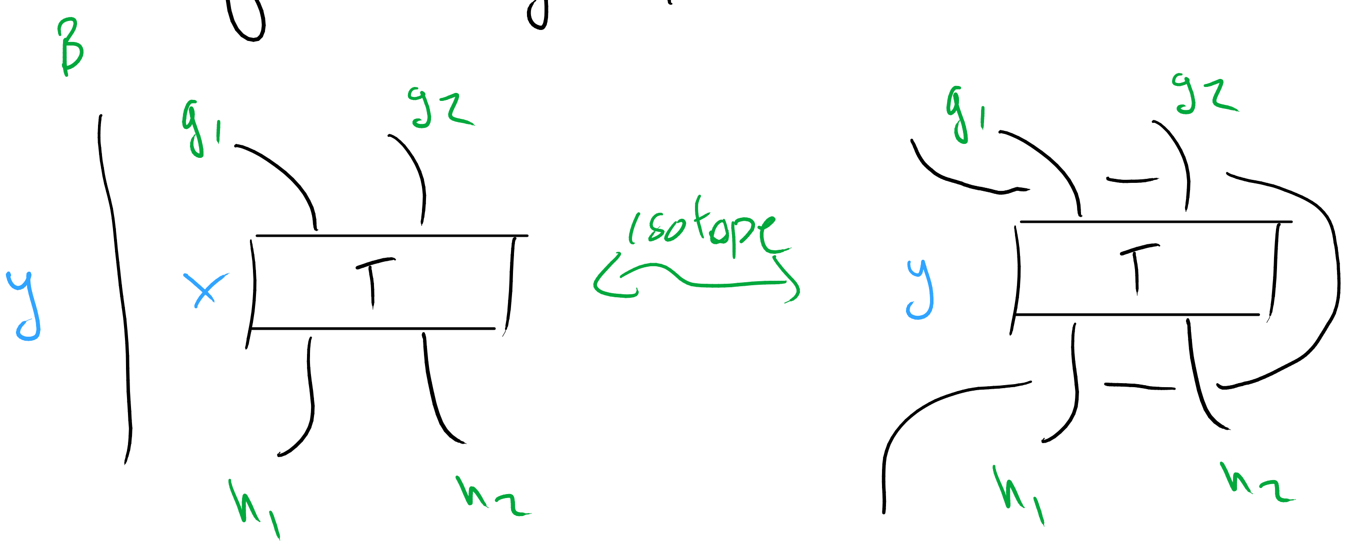
Region colors really are needed to get this to work:



are not equal in general: are not even elements of same space!

Gauge equivalence ~ braiding

If $x = yB$, $B \in SL_2(\mathbb{C})$ then



Can relate values using braiding action on boundary strands

If T is a link then indep of shadow

Thm [me, part with N. Reshetikhin]

There is a quantum invariant $\mathcal{Z}_N(T, \rho)$ of tangles that is a quantization* of $\mathcal{I}(T, \rho)$

*in an algebraic sense

When ρ is trivial get Kashaev's invariant

Related to (hopefully)

- $SL_2(\mathbb{C})$ quantum Chern–Simons (WRT is usually $SU(2)$)
- volume conjecture

Construction uses $\quad \xi = e^{\pi i/N}$

- cyclic $\mathcal{U}_\xi(\mathfrak{sl}_2)$ modules $E^N, F^N \neq 0$
- parametrized by same coordinates as octahedral decomposition
- need to use same diagrams as for $\mathcal{I}$ to get it to work

Reason 2 you should care: Zesting

original construction uses ordinary A-graded categories (A = abelian grp)

recent work [Me, Delaney] uses $\mathcal{T}ang(A, A)$

Zesting [Delaney, Galindo, Plavnik, Rowell, Zhang]

$\mathcal{C}$ = modular* tensor category

* can be weakened

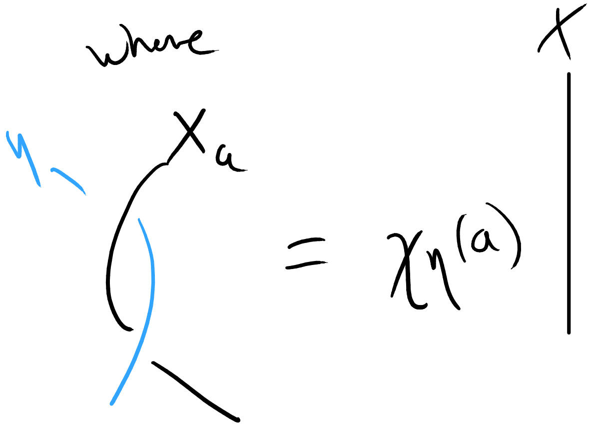
A = universal grading group of $\mathcal{C}$

maximal A with $\mathcal{C}$ faithfully A-graded. Concretely:

$\chi: \mathrm{Inv}(\mathcal{C}_e) \to \hat{A}$, $\eta \mapsto \chi_\eta$

where

$|X_a| = a$



$\zeta = (\lambda, \nu, t, \varepsilon)$ zesting data

- $\lambda: A^2 \to \mathrm{Inv}(\mathcal{C}_e)$ a symmetric 2-cocycle (monoidal structure)
- $\nu(a,b,c): \lambda(a,b) \otimes \lambda(ab,c) \to \lambda(b,c) \otimes \lambda(a,bc)$ realized by
- $t: \lambda(a,b) \to \lambda(b,a)$ (braiding)
- $\varepsilon(a) \in \Bbbk^\times$ (pivotal structure)

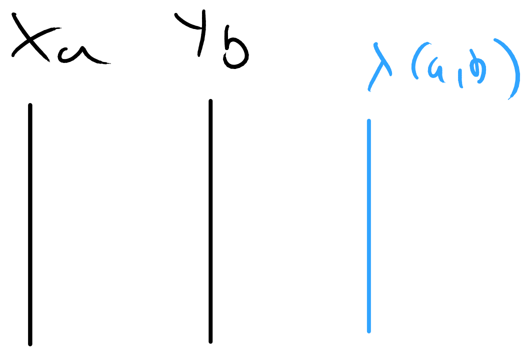
Thm [DGPRZ]

If these satisfy compatibility conditions,

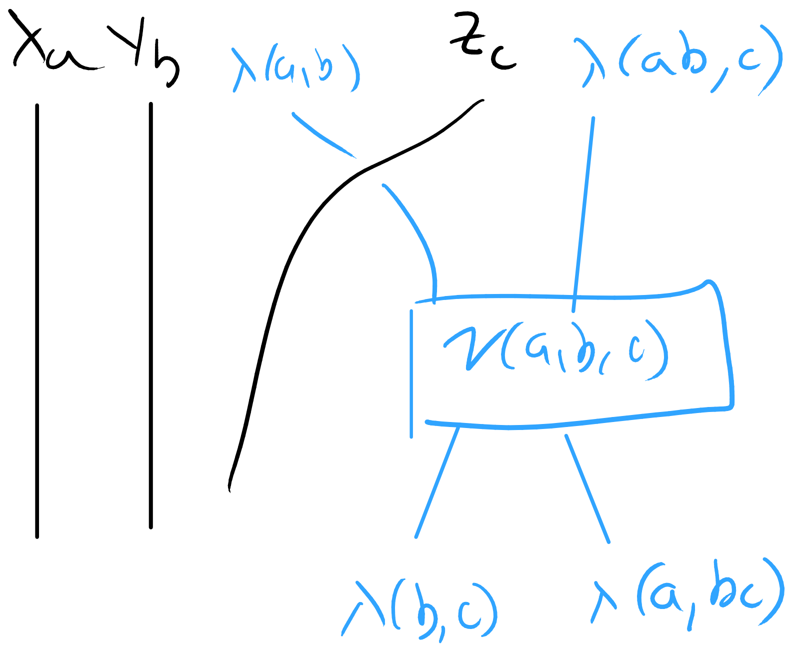
$$\mathcal{C} \underset{\text{zest}}{\rightsquigarrow} \mathcal{C}^\zeta$$ new MTC

$\mathcal{C}^{\zeta}$ has same objects,

new $\otimes$ product $\otimes_{\zeta}$

$X_a \otimes_{\zeta} Y_b :=$ 

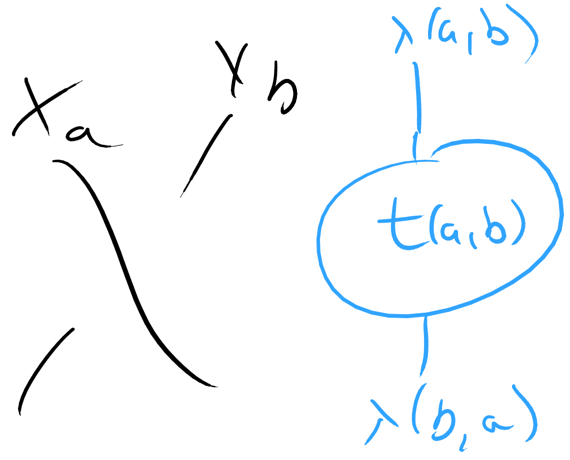
$$(X_a \otimes_{\zeta} Y_b) \otimes_{\zeta} Z_c \xrightarrow{\alpha_{\zeta}} X_a \otimes_{\zeta} (Y_b \otimes Z_c)$$



associators

New braiding

$$\beta^{\xi}_{X_a, Y_b} : X_a \otimes_{\xi} Y_b \to Y_b \otimes_{\xi} X_a$$



ξ is involved in duality

tangle colored by objects of $\mathcal{C}$ $\longrightarrow$ $\mathcal{F}_{\mathcal{C}}(T)$ RT inv

$\updownarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \updownarrow ?$

coloring by objects of $\mathcal{C}_{\zeta}$ $\longrightarrow$ $\mathcal{F}_{\mathcal{C}^{\zeta}}(T)$ RT inv

Question! How are $\mathcal{F}_{\mathcal{C}}(T)$ and $\mathcal{F}_{\mathcal{C}^{\zeta}}(T)$ related?

Thm [Delaney, Kim, Plavnik] If coloring is A-homogeneous and T is a link, there is a scalar $J(\mathcal{C}, \zeta, L)$ with $\mathcal{F}_{\mathcal{C}^{\zeta}}(L) = \mathcal{F}_{\mathcal{C}}(L)\, J(\mathcal{C}, \zeta, L)$

What is this scalar? Is it an RT invariant?

Does this work for tangles?

Thm [Me, Delaney] Yes, but you need to use an extra grading.

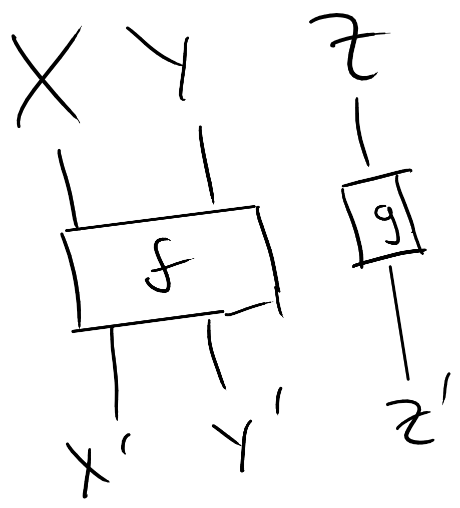
to appear

Can define functor $J_2 : \mathrm{Tang}(A,A) \to \mathrm{Inv}(\mathcal{C}_e)$

for $G = A$

$X = A$ acting on A

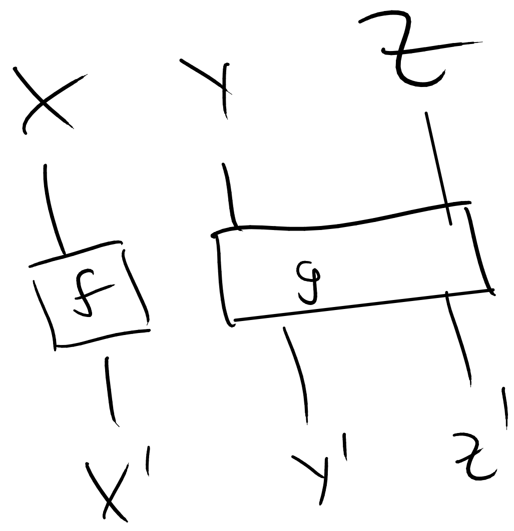
Convention When using graphical calculus for a non-strict $\mathcal{C}$, always left-associate



means

$$(X\otimes Y)\otimes Z \longrightarrow (X'\otimes Y')\otimes Z$$

$$f\otimes g$$



means

$$(X\otimes Y)\otimes Z \longrightarrow (X'\otimes Y')\otimes Z$$

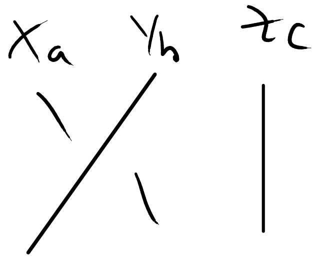
$$\alpha (f\otimes g)\alpha^{-1}$$

Goal Find a functor $J_z : \mathrm{Tang}(A) \to \mathcal{C}$ so that

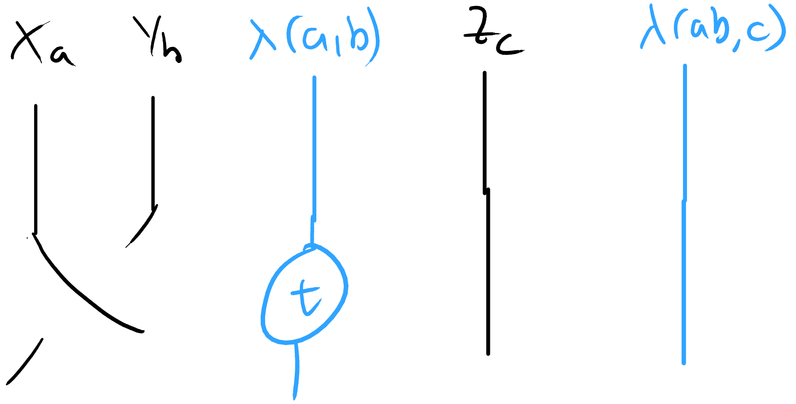
$\mathcal{J}_{e^z}(T)$ is a product of $\mathcal{J}_e(T)$ and $J_z(T)$

(in some sense)

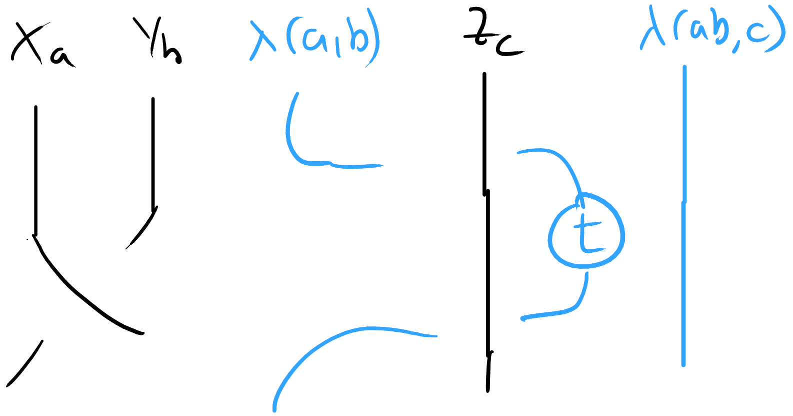
We will see that we need to use $\mathrm{Tang}(A, A)$ as the domain



$\xrightarrow[\text{zest}]{}$

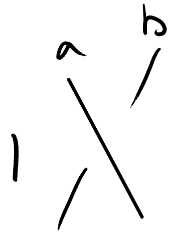
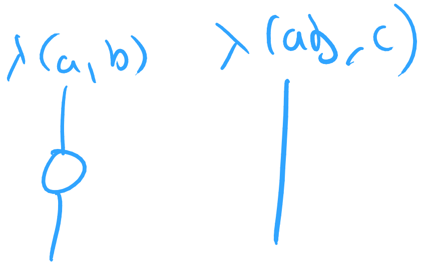


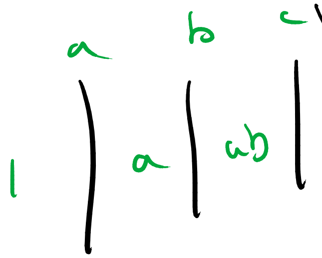
$=$



So we define

$$J_2(1;a,b,c) = \lambda(a,b)\otimes\lambda(ab,c)$$

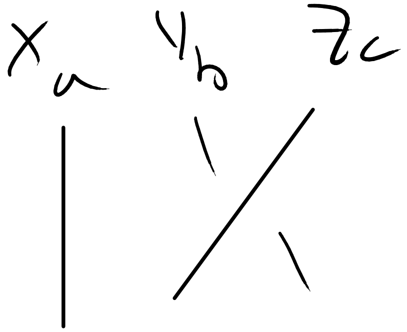
$J_2\Big($  $\Big) =$ 

$J\Big($  $\Big) = J_2(1;a)\otimes J_2(a;b)\otimes J_2(ab,c)$

$$= \lambda(1,a)\otimes\lambda(a,b)\otimes\lambda(ab,c)$$

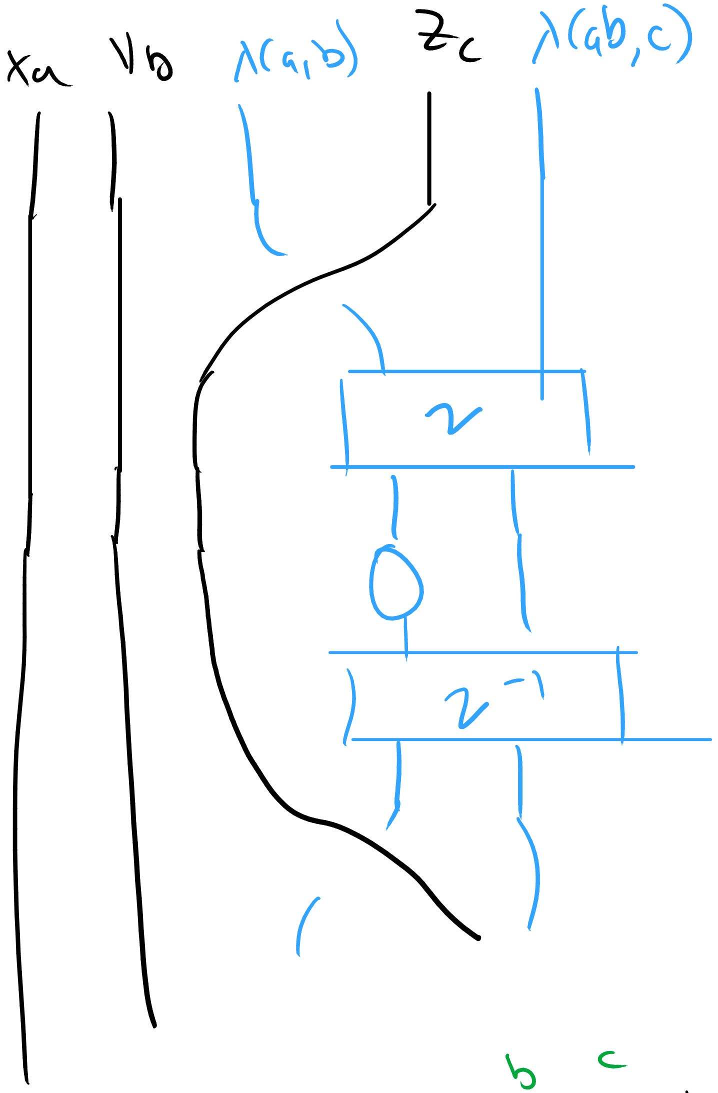
$$= \lambda(a,b)\otimes\lambda(a,c)$$

we need the second grading

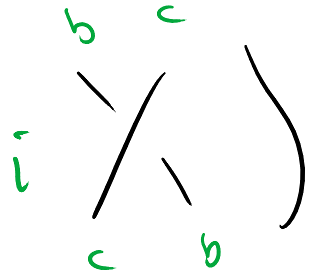


$\rightsquigarrow$ zest

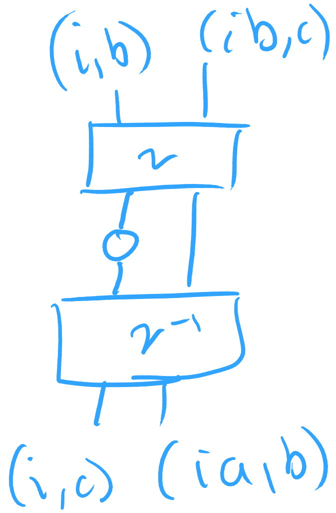
(convention: always left-associate)



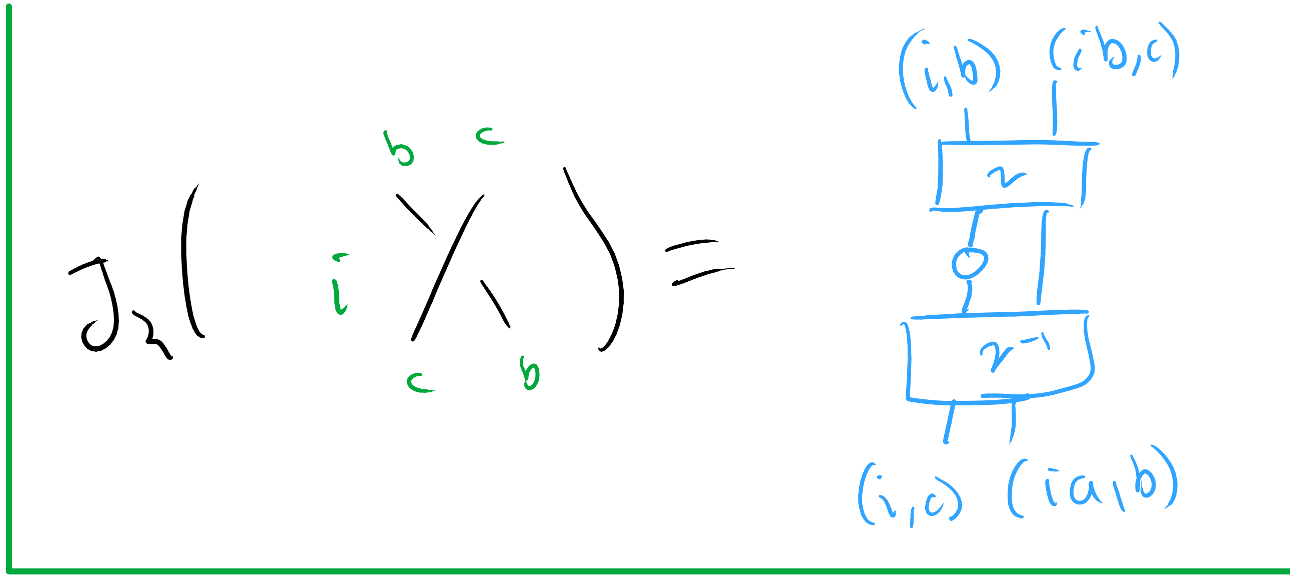
$$
\begin{tikzcd}
(X_a \otimes Y_b) \otimes Z_c \arrow[d, "\alpha"] \\
X_a \otimes (Y_b \otimes Z_c) \arrow[d, "\beta"] \\
X_a \otimes (Z_c \otimes Y_b) \arrow[d, "\alpha^{-1}"] \\
(X_a \otimes Z_c) \otimes Y_b
\end{tikzcd}
$$

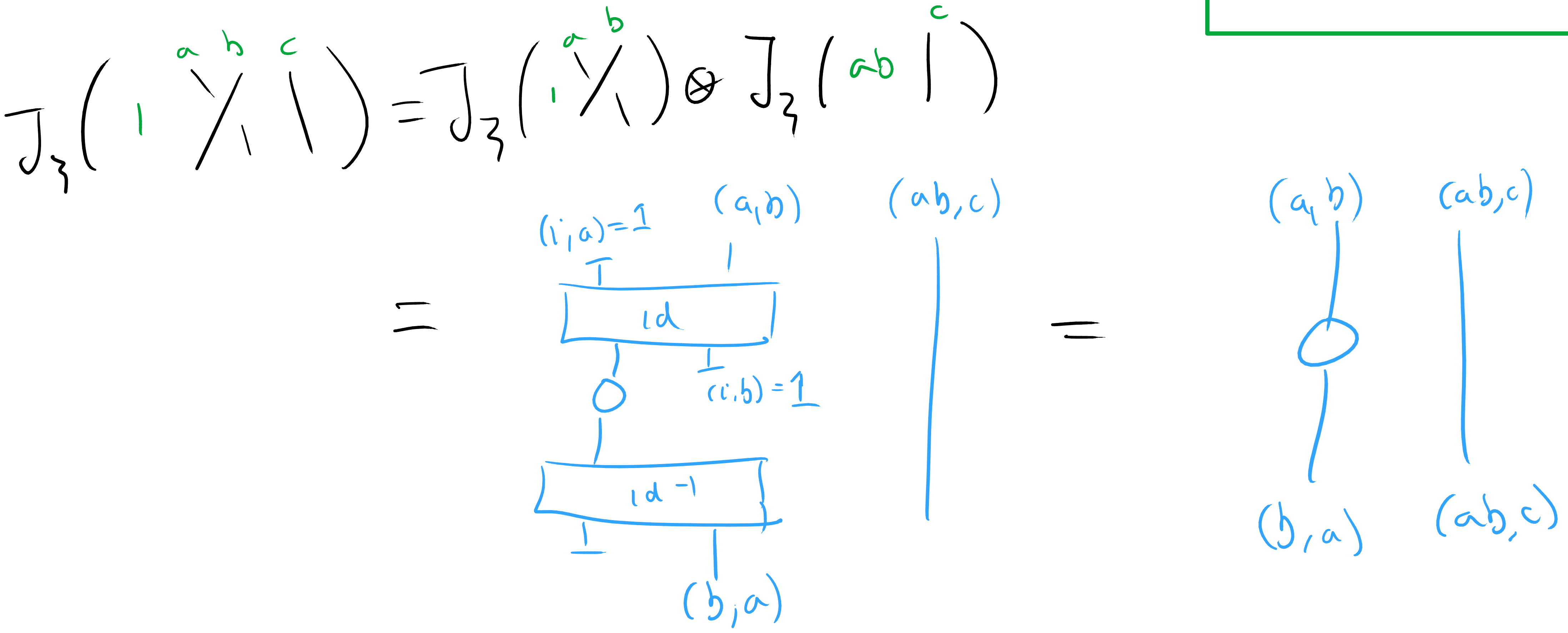
So define $J_2\Big($ 

$\Big) =$



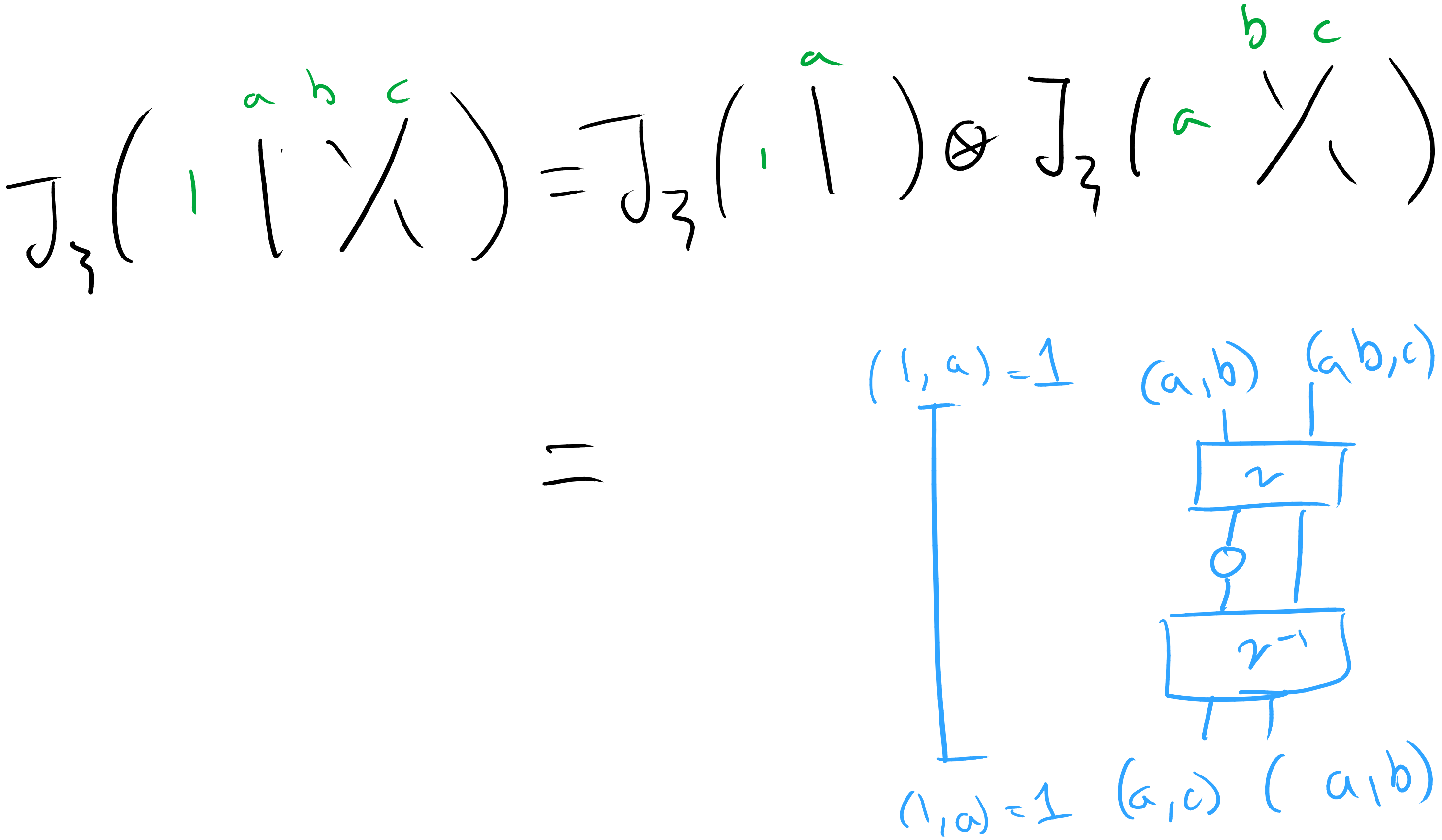
Gives the right answer!

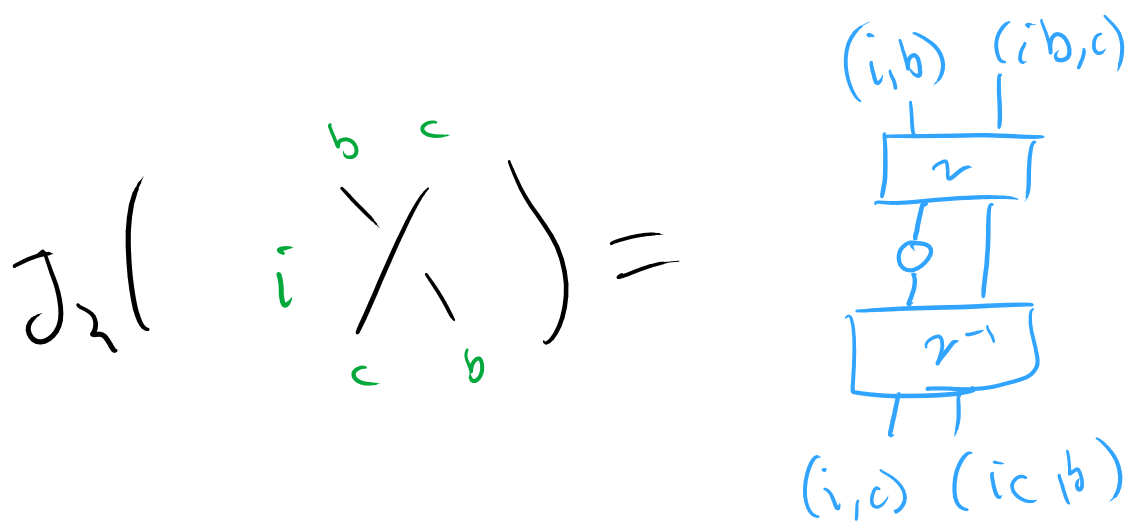




right answer!

Gives the right answer:



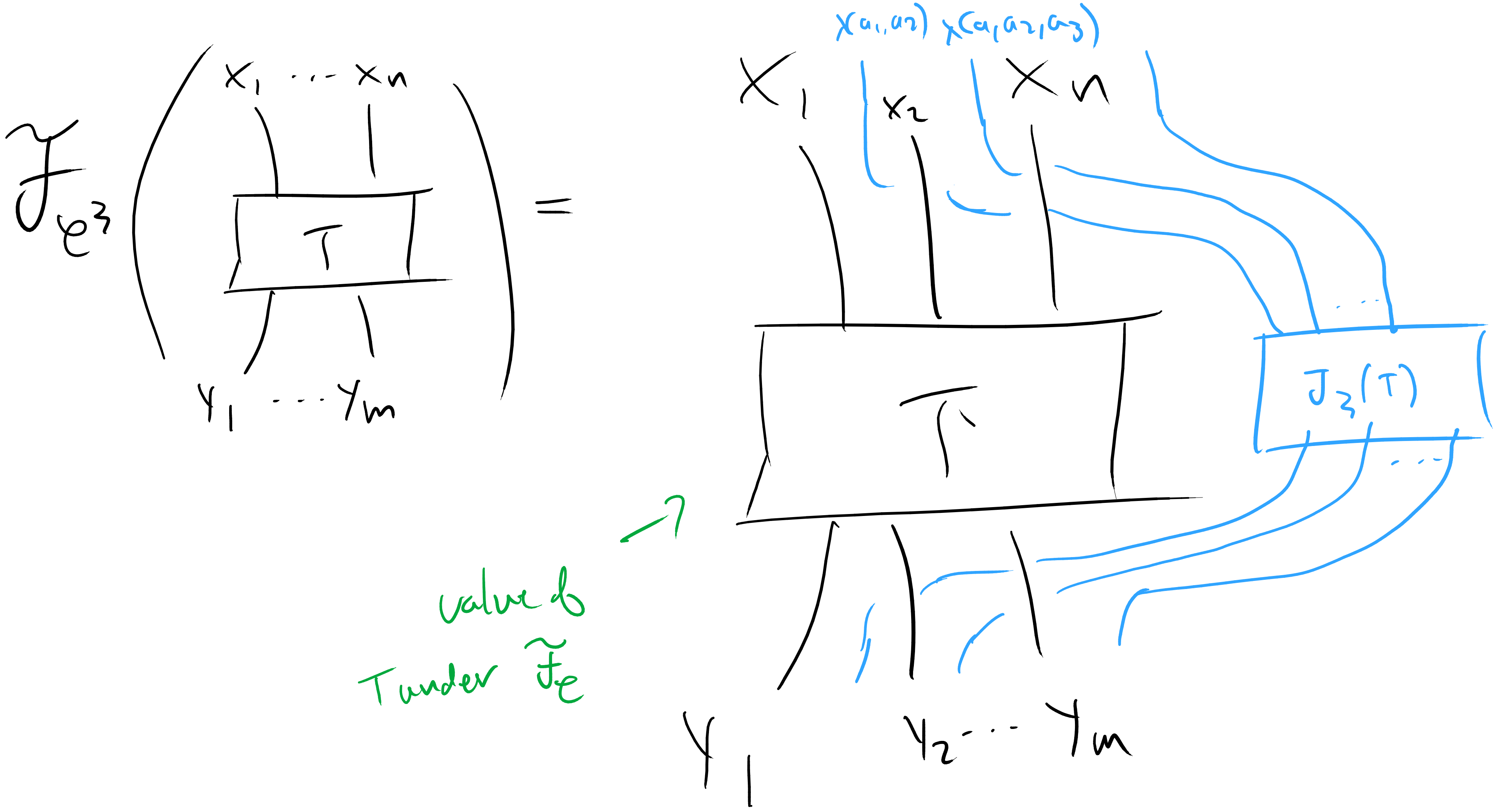


right answer!

Takeaway: associators in $\mathcal{C}^{?}$ mean you have to keep track of A-gradings of earlier terms in products

Exactly done by functor J_2 and $\mathrm{Tang}(A, A)$

Thm [Me, Delaney] For any homogeneous $\mathcal{C}$-colored tangle T,



we call this the warp product of $\mathcal{J}(T)$ and $J_3(T)$

where $J_3 : \mathrm{Tang}(A,A) \longrightarrow \mathrm{Inv}(\mathcal{C}_e)$ is well-defined and can be computed in polynomial time.

Problems/Questions

1. a) How does J_3 relate to condensed fiber product characterization of zesting?

   ↑ Julia Plavnik's talk?

   b) Is the image of J_3 a 2-category $\mathcal{B}_3$ in a useful way?

   c) Is $\mathcal{C}^2$ a fiber product of $\mathcal{B}_3$ and $\mathcal{C}$?

Problems / Questions

2. What is the right definition/perspective for (G,X)-graded 2-cats?

I know how to define $\mathrm{Tang}(G,X)$: what are targets?

I can write down some axioms but what are right ones?

have not yet

Problems/Questions

3. How does this relate to Sözer–Virelizier invariants

↑ Kürşat Sözer's talk

from homotopy 2-types? We have a G and an action on X, and in both examples X is a group too.

Maybe!

0-morphisms	X	π_2
1-morphisms	G	π_1
2-morphisms	?	?

related question

What is π_2 of a tangle complement?

$\pi_2(S^3 \setminus L) = 0$ for L a link.

Has to do w/ gauge invariance?