Surgery calculus for 3-manifolds with hyperbolic structures

by

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To obtain 3-manifold M:

- cut out torus nbhds of link $L \subset S^3$
- glue back in along (p,q) curve:

$$p\,m + q\,\ell = 0 \text{ in } M$$

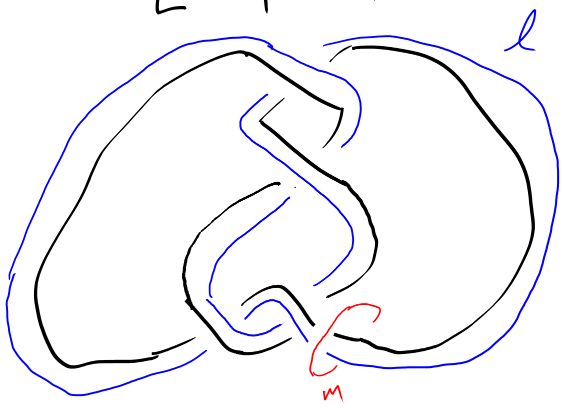
meridian, longitude

- need to specify p/q for each cpt. of L

Thm: <u>All</u> compact oriented M occur this way.

Thm: L, L' give same $M \iff$ related by Kirby–Fenn–Rourke moves

L = 4, knot



(Want ℓ to be 0-framed. This one is. If not, adjust.)

Bonus: This L is hyperbolic, so most p/q surgeries are too. Come from deformations of complete hyperbolic structure.

Hyperbolic = uniform curvature -1 metric

Thurston: There are lots of interesting hyperbolic manifolds.

Alg description:

$\{\text{hyp structure on } M\}$

$\longleftrightarrow$

$\{\rho : \pi_1(M) \to PSL_2(\mathbb{C})\}/\text{conjugation}$

Not necessarily complete! That means ρ is discrete + faithful.

Model: $\mathbb{H}^3 = \mathbb{C} \times (0, \infty)$

$\partial \mathbb{H}^3$ at ∞ is $\mathbb{C} \cup \infty = \hat{\mathbb{C}}$

Riemann sphere

with Euclidean metric

$\text{Isom}(\mathbb{H}^3) = \text{Isom}(\hat{\mathbb{C}})$

with hyp. metric

$= PSL_2(\mathbb{C})$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}$$

fractional linear transf's.

How do we do this for surgery presentations?

$\pi_1(S^3 \setminus L) =$ generated by meridians

one x_i per arc,
Conjugation relations at crossings

$\rho : \pi_1(S^3 \setminus L) \to PSL_2(\mathbb{C})$
pick $\rho(x_i)$ for each, satisfying rels

$\pi_1(M) = \pi_1(S^3 \setminus L) / \langle (p,q) \text{ curves to } 1 \rangle$

so need to write longitudes in terms of the x_i (not so bad) <u>and</u> check

$\rho(x_i)^p \rho(l_j)^q = 0$

Problem: This is too hard to do in practice.

4 vars $\begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$ per x_i, lots of complicated relations, ...

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There <u>is</u> a better way!

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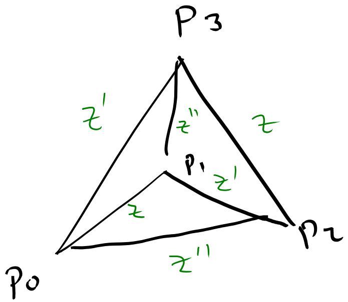
Thurston gave another description!

Triangulate $S^3 \setminus L$ with ideal tetrahedra with verices in $\hat{\mathbb{C}} = \partial \mathbb{H}^3$ on cpts of L



Cross-ratio $[p_0 : p_1 : p_2 : p_3]$

$= [z : 1 : \infty : 0]$ for $z \in \mathbb{C} \setminus \{0,1\}$

act by $PSL_2(\mathbb{C})$

To get hyperbolic structure, make sure $\prod_j z_j^{r_j} = 1$ at each edge.

$z =$ shape parameter

= complex dihedral angle

$z' = \frac{1}{1-z}$, $z'' = 1 - \frac{1}{z}$ other edges

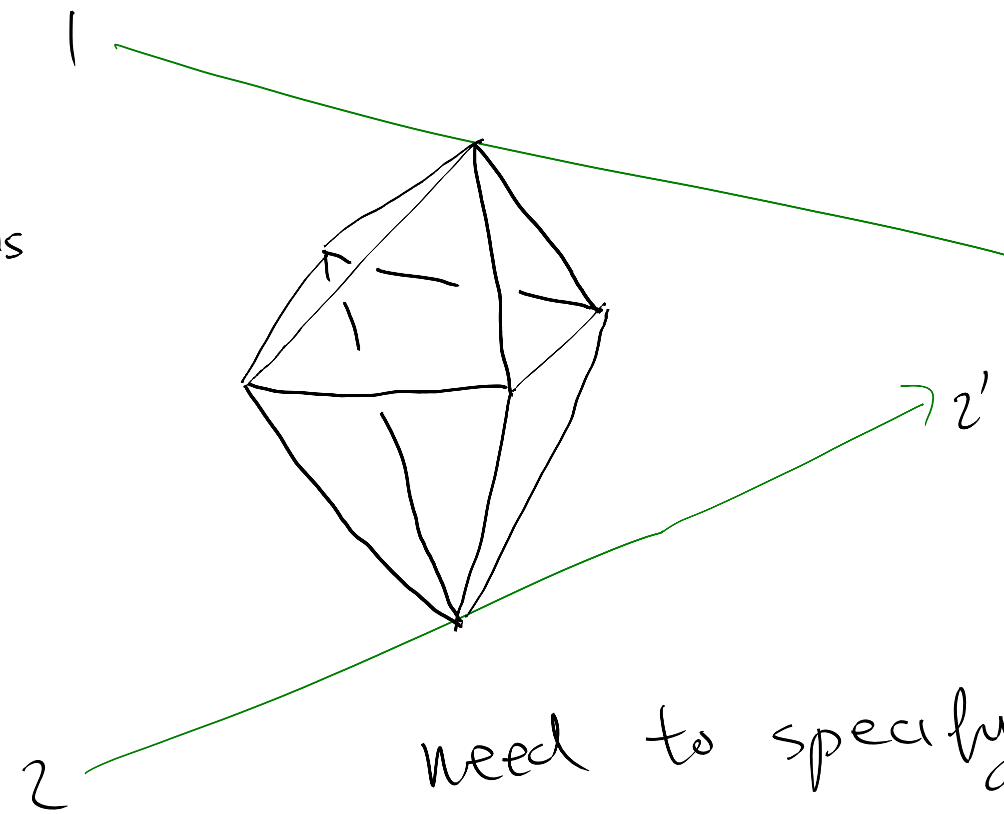
z_j = <u>geometric</u> coordinates for hyperbolic structures

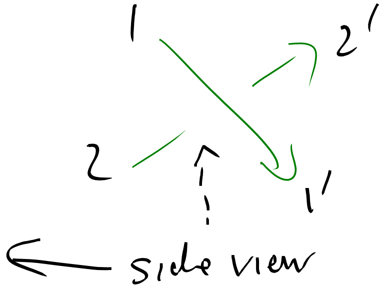
How to relate to surgery diagrams?

Pick standard ideal triangulation coming from a link diagram

octahedron at each crossing

(technically, yields 2 extra ideal points)





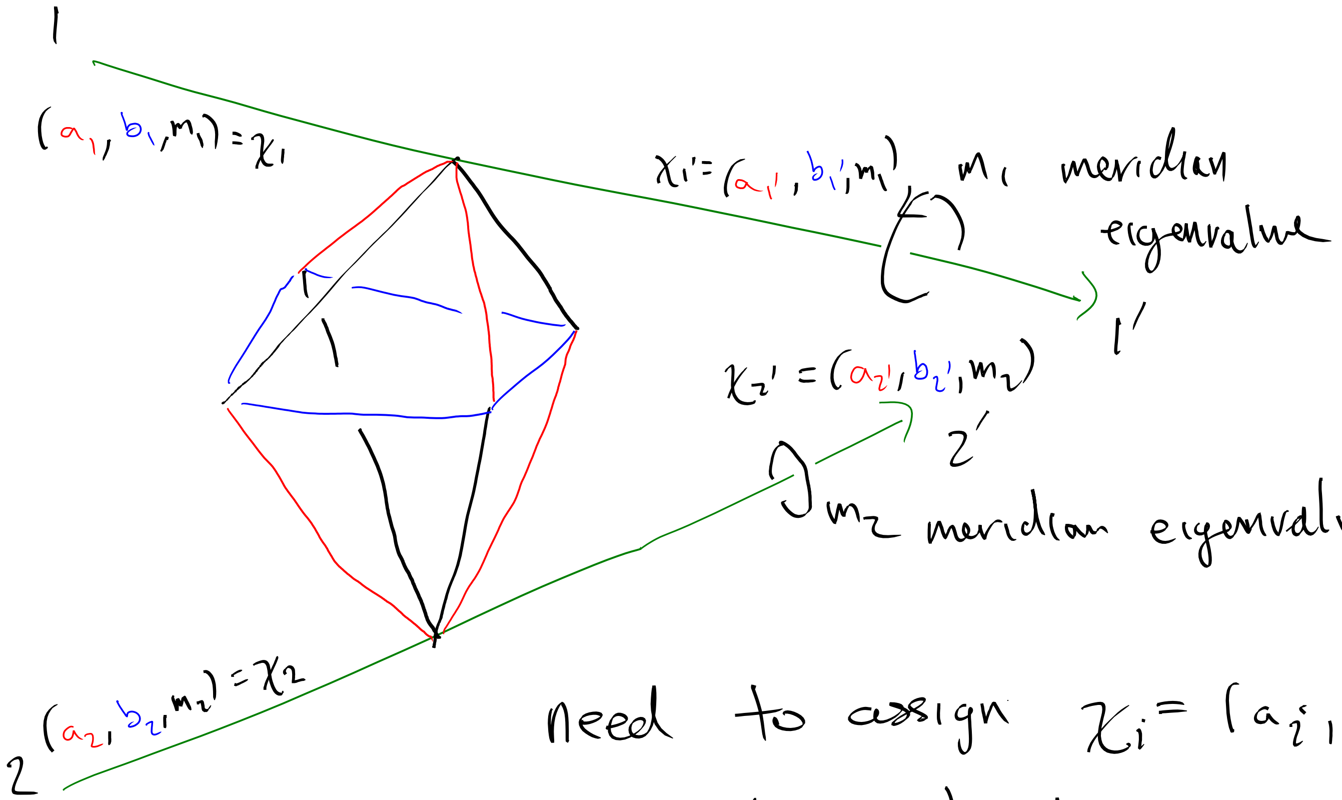
need to specify shapes at edges.

My preferred way:

red edges = a-variables

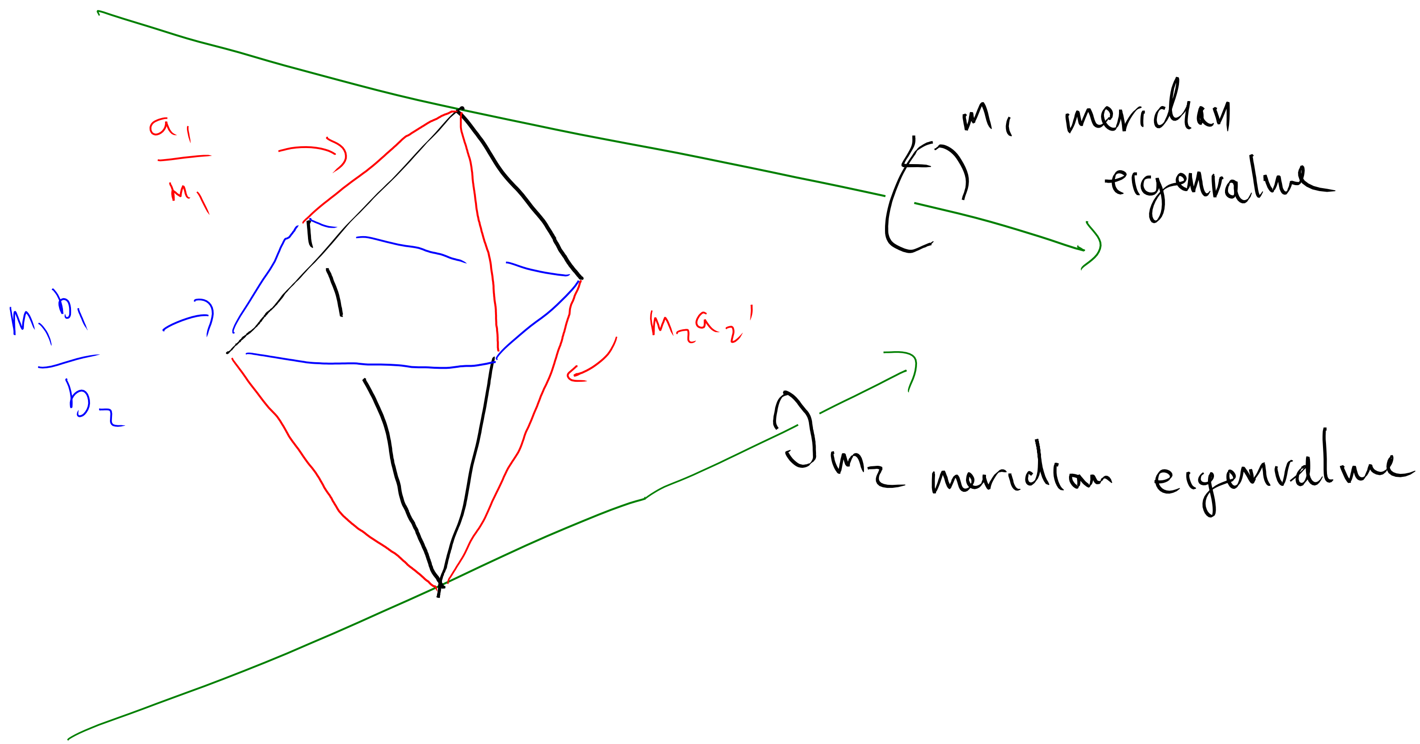
blue edges = b-variables

black edges are specified by these



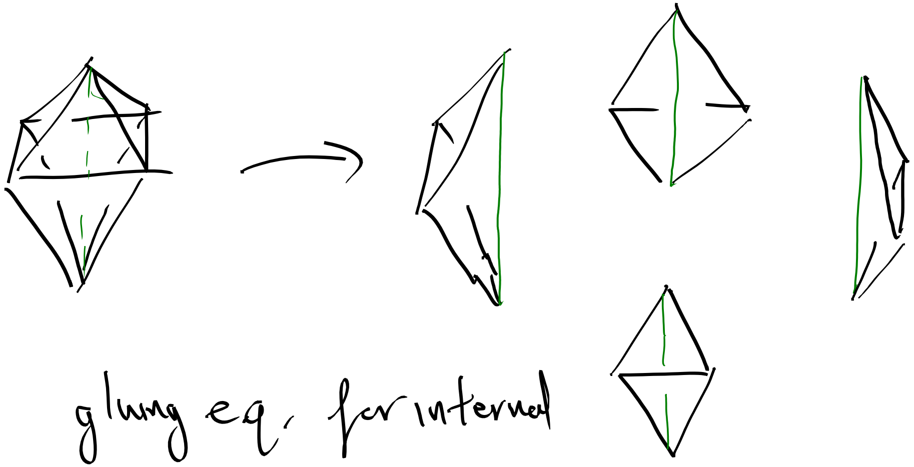
Need to assign $\chi_i = (a_i, b_i, m_i)$ to each diagram segment

Ex:

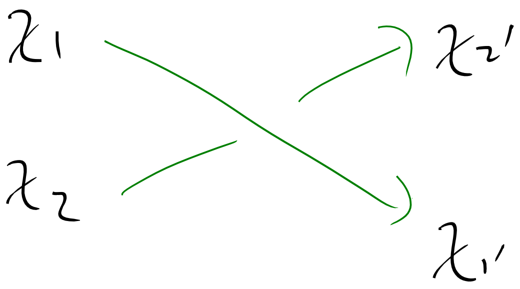


$\frac{a_1}{m_1}$
$\frac{m_1 b_1}{b_2}$
$m_2 a_2'$
m_1 meridian eigenvalue
m_2 meridian eigenvalue

When subdividing get internal conditions on χ_i



gluing eq. for internal edge gives conditions on the χ_i at each crossing



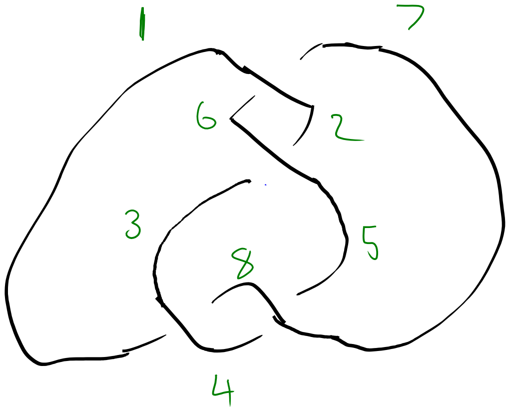
$$B(\chi_1, \chi_2) = (\chi_2', \chi_1')$$

"shape biquandle"

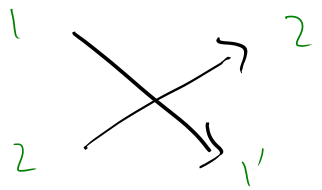
χ_i = "shapes"

Ex: To give hyp structure on $S^3 \setminus 4_1$, specify

$\chi_1, \dots, \chi_8$



with $B^{\pm 1}(\chi_1, \chi_2) = (\chi_2', \chi_1')$ at crossings.



In practice:

- specify m ahead of time
- can eliminate $\{a_i\}$ for $\{b_i\}$ or vice-versa

To upgrade to surgery:

Fact: $\underline{m}$ and $\underline{\ell}$ commute, so in some basis

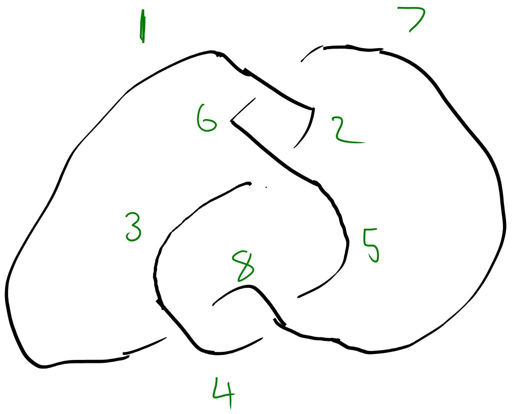
$$\rho(\underline{m}) \sim \begin{bmatrix} m & * \\ 0 & m^{-1} \end{bmatrix}, \qquad \rho(\underline{\ell}) \sim \begin{bmatrix} \ell & * \\ 0 & \ell^{-1} \end{bmatrix}$$

$$\rho(\underline{m})^p \rho(\underline{\ell})^q = 1 \iff m^p \ell^q = 1.$$

Much easier to solve!

give $\underline{\text{decorated}}$ or $\underline{\text{augmented}}$ representation

Octahedral coords. specify m $\underline{\text{and}}$ ℓ. ℓ comes from explicit product!



$$\leadsto \quad \ell = b_1^{-1} b_2 b_3^{-1} b_4 b_5^{-1} b_6 b_7^{-1} b_8$$

Conclusion: $\{\chi_i\}$ are a better way to parametrize hyperbolic structures on links and on surgery diagrams.

Other benefits:

- can get matrices of ρ back from $\{\chi_i\}$
- can directly compute volume $+ i$ Chern-Simons $=$ complex volume
  - why? Ptolemy coordinates $\rightsquigarrow$ flattenings for free!
- $\chi =$ central characters of $\mathcal{U}_\xi(\mathfrak{sl}_2)$ at $\xi = \exp(\pi i/N)$

Huh??

My real motivation (although stuff before was nice too)

(Joint w N. Reshetikhin)

Thm: There is a <u>quantized complex volume</u> $\mathcal{V}_N(L, \rho, s)$ "log-decoration"

$\mathcal{V}_N$ is sort of like complex volume $\mathcal{V} = \mathrm{vol} + i\,\text{Chern-Simons}$

$\mathcal{V}_N$ is sort of like Nth colored Jones/ Kashaev inv.

$(\mathcal{V}_N(L, 1, 0) =$ Nth Kashaev invariant$)$

trivial ρ

To define/understand/compute $\mathcal{V}_N$, need to use $\{\chi_i\}$