Quantum hyperbolic topology

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Acknowledgements

- Thanks to Seonhwa Kim and Jinsung Park for inviting me to give this talk.
- Many people have contributed to the mathematics I will discuss. I have tried to cite them all, but I may have gaps. My apologies!
- Later I will mention some highest-weight modules. I have tried to get the conventions to match [Bla+16] but I may not have: look at their paper for the right ones.

Plan of the talk

1. Reminders on TQFT (Topological Quantum Field Theory)
2. Extension to geometric (quantum) field theory
3. An abelian example: the BCGP invariant
4. Towards nonabelian $SL_2(\mathbb{C})$-field theory
5. Connections to hyperbolic topology

Topological field theories

Topological field theories

Geometric field theories

Abelian $SL_2(\mathbb{C})$-field theory

Towards nonabelian $SL_2(\mathbb{C})$-field theory

TQFT

- A $d+1$ dimensional TQFT $\mathcal{F}$ is a way of assigning manifold invariants that can be cut into pieces:
    - $(d+1)$-manifolds are assigned complex numbers $\mathcal{F}(M)$
    - d-manifolds are assigned vector spaces $\mathcal{F}(X)$
    - cobordisms $\partial M = \overline{X} \coprod Y$ are linear maps $\mathcal{F}(M) : \mathcal{F}(X) \to \mathcal{F}(Y)$
- Formally: Let Cob_d be the category whose

  objects are oriented d-manifolds
  morphisms are oriented cobordisms between them

  and Vect be the category with

  objects $\mathbb{C}$-vector spaces
  morphisms linear maps

  Then a $d+1$ dimensional TQFT is a functor $\mathcal{F} : \mathrm{Cob}_d \to \mathrm{Vect}$.
- Both categories are monoidal with duals and $\mathcal{F}$ should respect these structures.

Cutting and pasting

- Say we cut M into two pieces $N_1 \cup N_2$ along X.
- Since $\mathcal{F}(\emptyset) = \mathbb{C}$ is monoidal unit, we get ingredients:
  - $\mathcal{F}(N_1) : \mathbb{C} \to \mathcal{F}(X)$ (vector)
  - $\mathcal{F}(N_2) : \mathcal{F}(X) \to \mathbb{C}$ (covector)
- Composition

$$\mathcal{F}(N_2)(\mathcal{F}(N_1)) = \mathcal{F}(M) \in \mathbb{C}$$

  is evaluating vector against dual vector
- More generally, can compute $\mathcal{F}(M)$ by cutting M into simple pieces N_j then composing resulting tensors.

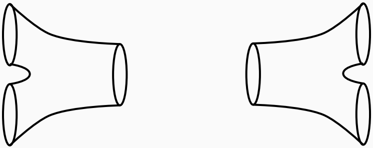
Example: $d = 1$

Say we want to define a $1+1$ dimensional TQFT.

- Only object in Cob_1 is S^1, so need a vector space $A = \mathcal{F}(S^1)$.
- Cobordisms will be maps between tensor powers of A and A^*
- For example, depending on orientation the disk D^2 is a cobordism $b : \emptyset \to S_1$ or $d : S^1 \to \emptyset$
- Then $\mathcal{F}(b) : \mathbb{C} \to A$ is a chosen vector and $\mathcal{F}(d) : A \to \mathbb{C}$ is chosen covector.

Example: $d = 1$

More interesting cobordisms come from pairs of pants.



- Left is a map $A \otimes A \to A$, right is a map $A \to A \otimes A$.
- By using topological relations, there are compatibility conditions on these.
- Turn out to make A into a Frobenius algebra

$d = 2$

- In higher dimensions, much more complicated, because manifolds are much more complicated.
- We mostly focus on $d = 2$, so we assign vector spaces to surfaces and complex numbers to closed 3-manfiolds.
- Famous example: the Witten-Reshetikhin-Turaev theory is a $2+1$ dimensional TQFT

Witten's version of WRT

Definition ([Wit89])

For a flat $\mathfrak{su}_2$ connection A on M, consider Chern-Simons invariant as a Lagrangian

$$\mathcal{L}(A) = \frac{1}{4\pi}\int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)$$

Then the path integral

$$Z(M) = \int \exp(ik\mathcal{L}(A))\,\mathcal{D}A$$

over all connections A gives value of a TQFT via $\mathcal{F}_k(M) = Z(M)/Z(S^3)$. Integer k is level.

- Can extend to case where M has an embedded link L
- This is not mathematically rigorous
- However, can use physical arguments to determine how $Z(M)$ changes under surgery on L, allowing computation

Reshetikhin-Turaev's version of WRT

Pick framed link L in S^3 representing M via Dehn surgery.

- For any labeling of components L_j of L by modules V_j of quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$, use R-matrix to construct invariant $\mathcal{F}(L; \{V_j\})$. Jones polynomial is a special case of these.
- When $q = \zeta$ is root of unity (order is related to level k) can get modular category of $\mathcal{U}_\zeta(\mathfrak{sl}_2)$-modules with special properties
- By taking weighted sum of *all* labellings of L by modules, get invariant $\mathcal{F}_k(M)$ of M.
- Physical arguments identify $\mathcal{F}_k(M)$ with Witten's $Z(M)/Z(S^3)$.
- $\mathcal{F}_k$ can be extended to a full $2+1$ TQFT.

Details in [RT91]. A good exposition is [BK00].

Manifolds with links

- In both cases, natural to extend to 3-manifolds M with an embedded link L (embedded copies of S^1)
- Related to the fact that these are extended TQFTs: can be extended to cobordism 2-category
- When $L = \emptyset$, recover usual invariant: $\mathcal{F}(M, \emptyset) = \mathcal{F}(M)$
- If M has nonempty boundary, we allow tangles that start or end on the boundary components
- Objects of our category are then surfaces with marked points where tangles can start or end

Geometric field theories

Topological field theories

Geometric field theories

Abelian $SL_2(\mathbb{C})$-field theory

Towards nonabelian $SL_2(\mathbb{C})$-field theory

Definition

Let G be a group (usually a Lie group) and M be a manifold. A G-structure is a representation $\rho : \pi_1(M) \to G$ considered up to conjugation.

Example

$G = \mathrm{PSL}_2(\mathbb{C})$ is the isometry group of hyperbolic 3-space, so hyperbolic structures on M are $\mathrm{PSL}_2(\mathbb{C})$-structures.

We focus on $G = \mathrm{SL}_2(\mathbb{C})$ and 3-manifolds. We think of a $\mathrm{SL}_2(\mathbb{C})$-structure as a generalized hyperbolic structure.

Why hyperbolic structures?

- Hyperbolic 3-manifolds are large, interesting class; these have $\mathrm{PSL}_2(\mathbb{C})$-structures that are discrete and faithful
- More generally, studying moduli space of $\mathrm{SL}_2(\mathbb{C})$-structures (character variety) on M gives important topological information about M
- For us, turns out to be convenient to use double cover $\mathrm{SL}_2(\mathbb{C})$ instead.

Geometric field theory

Definition

Cob_d^G is the category with

objects d-manifolds X with G-structures $\rho : \pi_1(X) \to G$

morphisms cobordisms M with G-structures $\rho : \pi_1(M) \to G$

To compose two morphisms we require that the G-structures match after our identification.

Definition

A G-field theory is a functor $\mathcal{F} : \mathrm{Cob}_d^G \to \mathrm{Vect}$ depending only on the conjugacy classes of the G-structures.

Turaev [Tur10] calls these homotopy quantum field theories with target $K(G,1)$.

Geometric 3-manifold invariants

Return to $G = \mathsf{SL}_2(\mathbb{C})$ and $d = 2$. If $\mathcal{F}$ is a $\mathsf{SL}_2(\mathbb{C})$-field theory in dimension $2 + 1$, then for each $\mathsf{SL}_2(\mathbb{C})$-structure ρ on a 3-manifold we get

$$\mathcal{F}(M, \rho) \in \mathbb{C}$$

If ρ' is conjugate to ρ then $\mathcal{F}(M, \rho) = \mathcal{F}(M, \rho')$.

Examples

Torsion

Reidemeister torsion $\tau(M, \rho)$ twisted by ρ can be thought of as value of a GFT.

Later we will discuss how to extend this to a GFT (instead of just for closed M.)

Complex volume

Natural to consider hyperbolic volume and Chern-Simons invariant as parts of a complex volume

$$\mathrm{cVol}(M, \rho) = \mathrm{Vol}(M, \rho) + i\,\mathrm{CS}(M, \rho) \in \mathbb{C}/\pi^2 i\mathbb{Z}$$

At least for M with torus boundary, can cut and glue cVol [KK93].

Another perspective

Definition

We write $\mathfrak{X}_M$ for the character variety of M. Up to technicalities $\mathfrak{X}_M$ is the moduli space of $\mathrm{SL}_2(\mathbb{C})$-structures on M.

- Now $\mathcal{F}$ assigns each 3-manifold M a function $\mathcal{F}(M)$ on its character $\mathfrak{X}_M$.
- Much more powerful than a TQFT: instead of one number we get a function on an interesting algebraic variety!

Extracting simpler invariants

However, $\mathfrak{X}_M$ can be complicated. We might want something simpler. Ways to do this:

- Pick trivial structure $\rho_{\text{triv}} \in \mathfrak{X}_M$ with $\rho_{\text{triv}}(x) = 1$ for all x
- If M is hyperbolic, there is a *canonical* structure ρ_{hol} by Mostow rigidity. $\mathcal{F}(M, \rho_{\text{hol}})$ is a topological invariant of M for any GFT $\mathcal{F}$.
- Restrict to simpler part $\mathfrak{A}_M \subset \mathfrak{X}_M$, say ρ with abelian image.

Abelian $SL_2(\mathbb{C})$-field theory

A simpler example

- Constructing a full $\mathsf{SL}_2(\mathbb{C})$-field theory is hard!
- As a first step, let's instead restrict to $\rho : \pi_1(M) \to \mathsf{SL}_2(\mathbb{C})$ with abelian image. After diagonalizing, this means

$$\rho(x) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t \in \mathbb{C} \setminus \{0\}$$

  for every $x \in \pi_1(M)$.
- Can think of this as restricting to $\mathsf{GL}_1(\mathbb{C})$ subgroup of $\mathsf{SL}_2(\mathbb{C})$

Abelian representations

Definition

For M a 3-manifold with an embedded link L, write

$$\mathfrak{A}_{M,L} = \mathrm{H}^1(M \setminus L; \mathbb{C}/2\mathbb{Z}).$$

We think of $\mathfrak{A}_{M,L}$ as part of the character variety: if $\omega \in \mathfrak{A}_{M,L}$ and $x \in \pi_1(M \setminus L)$, then

$$\rho(x) = \begin{pmatrix} \exp(\pi i \omega(x)) & 0 \\ 0 & \exp(-\pi i \omega(x)) \end{pmatrix}$$

(Actually slightly more: $\omega(x)$ *logarithm* of eigenvalues of x)

The BCGP field theory

Theorem (Blanchet, Costantino, Geer, and Patureau-Mirand [Bla+16])

Pick an even integer $2r$, $r \not\equiv 0 \pmod 4$. For each (M, L, ω) there is an invariant

$$\mathbb{V}_r(M, L, \omega) \in \mathbb{C}$$

Furthermore, this invariant extends to a geometric quantum field theory on a category with

- ***objects*** *surfaces with embedded marked points and compatible classes ω*
- ***morphims*** *cobordisms between surfaces with embedded tangles between the points, again with compatible classes ω*

Special cases

Say $M = S^3$ and K is a knot. Then $\mathfrak{A}_{S^3,K} \cong \mathbb{C}/2\mathbb{Z}$, so ω is a single number λ. We see that

$$\mathbb{V}_r(S^3, K, \omega) = \nabla_r(K, \lambda)$$

is a function of λ.

Theorem

$\nabla_r(K, \lambda)$ *is a rational function in* $t = \exp(\pi i \lambda)$ *and agrees with the invariant of Akutsu, Deguchi, and Ohtsuki [ADO92].*

In particular, for $r = 1$ it is the Conway polynomial (normalized Alexander polynomial).

We interpret $\mathbb{V}_r$ as an extension of ADO to a field theory.

Special cases

Theorem

If $\omega = 0$, then $\mathbb{V}_r(S^3, L, 0)$ is the Kashaev invariant, the rth colored Jones polynomial of L evaluated at $q = \exp(\pi i/r)$.

This is the invariant appearing in the volume conjecture.

We interpret $\mathbb{V}_r$ as extending the Kashaev invariant to a field theory, because it also makes sense for manifolds other than S^3.

Why bother?

Some advantages over usual RT:

- Any TQFT gives mapping class group representations; for RT Dehn twists are finite-order and obviously not faithful
- Mapping class group representations of BCGP are infinite-order, so potentially faithful
- BCGP can distinguish some lens spaces that RT cannot

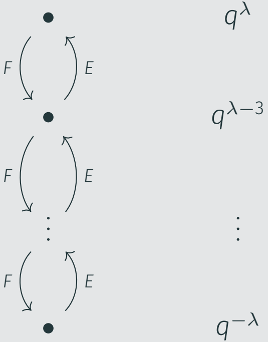
How to construct it

- As with RT, first step is invariants of framed links in S^3.
- Usual RT construction assigns representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ to components
- Now we use $\mathcal{U}_\xi(\mathfrak{sl}_2)$ at $\xi = \exp(\pi i/r)$
- Class ω assigns complex number λ_j to link component L_j (evaluate on meridian)
- We assign L_j a $\mathcal{U}_\xi(\mathfrak{sl}_2)$-module V_{λ_j} parametrized by λ_j
- Where do these come from?

Highest-weight modules

Fact

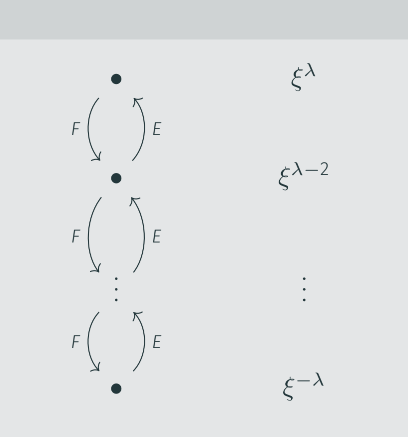
For q generic (not a root of unity) up to some signs any $\mathcal{U}_q(\mathfrak{sl}_2)$-module of dimension $\lambda + 1$ looks like V_λ given by



- Weights are eigenvalues of $K = q^H$ just like for usual $\mathfrak{sl}_2$
- Here we need highest weight λ to be an *integer*

q a root of unity

Now set $q = \xi = \exp(\pi i / r)$



- If highest weight $\lambda \in \{0, 1, \ldots, r-1\}$, get module V_λ of dimension $\lambda + 1$ specializing previous case
- If $\lambda \in \mathbb{Z}$ and $|\lambda| \geq r$, V_λ is no longer irreducible
- New modules: because $\xi^{2r} = 1$, can have modules V_λ of dimension r for any $\lambda \in \mathbb{C} \setminus \mathbb{Z}$

Representations of $\mathcal{U}_\xi(\mathfrak{sl}_2)$

$\lambda \in \{0, 1, \ldots, r-2\}$

- Modules V_λ are specializations of generic q case
- Non-vanishing quantum dimensions
- This part gives the modular category used in RT construction

$\lambda \in \mathbb{C} \setminus \mathbb{Z}$ or $\lambda = r-1$

- New, exotic behavior: non-integral highest-weights
- Vanishing quantum dimension
- These modules are sent to 0 in semi-simplification as part of RT construction

- Important case is V_{r-1}, used to construct Kashaev invariant.
- If $\lambda \in \mathbb{Z}$ and $\lambda \notin \{0, 1, \ldots, r-1\}$, much more complicated. We mostly avoid these modules.

Applying to BCGP construction

- To compute $\mathbb{V}_r(S^3, L, \omega)$ we assign component L_j with ω-value λ_j the module V_{λ_j}
- To get surgery invariant, there is a similar sum over all admissible labelings like in usual RT. (Roughly speaking, we sum over rth roots of $\exp(\pi i \lambda_j)$)
- One significant technical difficultly: because quantum dimension of V_λ vanishes, obvious construction vanishes. Need to use modified traces to fix this.
- For this reason BCGP is sometimes called a non-semisimple TQFT

Towards nonabelian $SL_2(\mathbb{C})$-field theory

Nonabelian holonomy

- The BCGP invariant is defined for $\rho : \pi_1(M) \to \mathsf{SL}_2(\mathbb{C})$ with *abelian image*
- Problem: geometrically interesting representations never have abelian image!
- For example, canonical holonomy rep ρ_{hol} of hyperbolic M is faithful, so never abelian

Our goal

Extend BCGP theory $\mathbb{V}_r(M, L, \omega)$ to $\mathrm{SL}_2(\mathbb{C})$-field theory. Corresponding quantum holonomy invariants are

$$\mathbb{F}_r(M, L, \rho, \omega) \in \mathbb{C}$$

Can think of this as a deformation or twisting of Kashaev/ADO invariants by ρ.

- In abelian case cohomology class ω determined ρ, plus logarithm of meridian eigenvalues
- Now ω is a similar choice of logarithm, needs to be compatible with ρ

$\mathbb{F}_r$ has not yet been defined in general. I want to explain what is known and discuss remaining obstacles.

Physical interpretation

- Recall that WRT theory $\mathcal{F}_k$ was quantum Chern-Simons theory with gauge group $\mathsf{SU}(2)$
- $\mathbb{F}_r$ should be closely related to quantum Chern-Simons with *noncompact* gauge group $\mathsf{SL}_2(\mathbb{C})$ [Guk05]
- Interesting in context of volume conjecture

The volume conjecture

Recall that $\mathbb{F}_r(S^3, L, \rho_{\text{triv}}, 0) = \mathbb{V}_r(S^3, L, 0)$ is the Kashaev invariant, equivalently the rth colored Jones polynomial at $q = \exp(\pi i/r)$.

Conjecture ([Kas97], [MM01])

For any hyperbolic knot K in S^3,

$$\lim_{r \to \infty} \frac{\log |\mathbb{F}_r(S^3, K, \rho_{triv}, 0)|}{r} = \frac{\operatorname{Vol}(K, \rho_{hol})}{2\pi}$$

where $\operatorname{Vol}(K, \rho_{hol})$ is the hyperbolic volume of the canonical holonomy representation ρ_{hol}.

Question

How does value at trivial representation know about the canonical hyperbolic structure?

The volume conjecture and GFT

In the context of $\mathrm{SL}_2(\mathbb{C})$-field theory, can at least split this into two conjectures:

Conjecture

$$\lim_{r\to\infty} \frac{\log|\mathbb{F}_r(S^3, K, \rho_{hol}, 0)|}{r} = \frac{\mathrm{Vol}(K, \rho_{hol})}{2\pi}$$

Conjecture

$$\lim_{r\to\infty} \frac{\log|\mathbb{F}_r(S^3, K, \rho_{triv}, 0)|}{r} = \lim_{r\to\infty} \frac{\log|\mathbb{F}_r(S^3, K, \rho_{hol}, 0)|}{r}$$

First seems plausible. Second is harder but Witten [Wit11] suggests physical reasons it might be true.

Links in S^3

First step: links L in $M = S^3$.

Theorem (Blanchet, Geer, Patureau-Mirand, and Reshetikhin [Bla+20])

Up to phase indeterminacy there is such an invariant

$$F_r(L, \rho, \omega) \in \mathbb{C}/\Gamma_{r^2}$$

where Γ_n is the group of nth roots of unity. Here ρ is any boundary non-parabolic representation.

Problem

Cannot make sense of sum of things in $\mathbb{C}/\Gamma_{r^2}$; need to fix this to use RT construction.

Boundary non-parabolic is also a problem (hyperbolic links are always boundary parabolic) but is easier to fix.

Defining the invariant

- Before, we used modules V_λ with highest weight $\lambda \in \mathbb{C}$
- These were unusual, but still had E^r and F^r acting by 0
- However, there are cyclic modules $V_{\chi,\lambda}$ where this is no longer true!
- Now they are parametrized by matrices

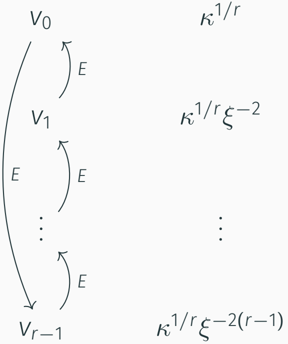
$$\begin{bmatrix} \chi(K^r) & -\chi(E^r) \\ \chi(K^r F^r) & \cdots \end{bmatrix} \in \mathsf{SL}_2(\mathbb{C})$$

  where χ is a character on a central subalgebra $\mathcal{Z}_0 \subset \mathcal{U}_\xi(\mathfrak{sl}_2)$ that appears at $q = \xi$.
- λ is related to action of Casimir on $V_{\chi,\lambda}$

Cyclic modules

- Cyclic modules are parametrized by character χ and "highest weight" λ, which is not really a highest weight anymore because $\ker E = 0$.
- Instead λ determines action of the Casimir element
- χ is related to value of holonomy ρ around the strand of a link
- λ is logarithm of eigenvalues of the holonomy

$$\chi(K^r) = \kappa, \chi(E^r) = \epsilon$$



$$E \cdot v_k = v_{k-1}$$

$$E \cdot v_0 = \epsilon v_{r-1}$$

The braiding

- Key step in RT link invariants is defining the braiding $V \otimes W \to W \otimes V$
- Usually determined by action of universal R-matrix on $V \otimes W$:

$$\mathsf{R} = q^{H \otimes H/2} \sum_{n=0}^{\infty} c_n E^n \otimes F^n \in \mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2}$$

- For ordinary RT and BCGP, E and F act nilpotently so action of **R** converges
- Problem: for cyclic modules action of **R** diverges

Fixing the braiding

- To fix this, Kashaev and Reshetikhin [KR04; KR05] suggest looking at conjugation action of R on $\mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2}$, which still makes sense at $q = \xi$
- Can use this to uniquely characterize braiding on modules
- However, does not fix normalization or give explicit formula
- Consequences:
    1. $\mathbb{F}_r$ has indeterminate phase
    2. Very difficult to compute values
    3. Hard to relate to geometry

A special case

In simplest nontrivial case, something can be said:

Theorem (Me, [McP22a; McP22b])

For any link L in S^3,

$$F_2(L, \rho, \omega) F_2(\overline{L}, \overline{\rho}, \omega) = \tau(S^3 \setminus L, \rho)$$

where $\overline{L}$ is the mirror image and $\tau(S^3 \setminus L, \rho)$ is the Reidemeister torsion twisted by ρ.

This is the natural generalization of the usual construction of the Alexander polynomial as a quantum invariant.

Proof.

Instead of computing braiding directly, use quantum doubles to give a different characterization that it easier to work with. □

Understanding the braiding

We want to understand the braiding better in general.

Theorem (Me, Reshetikhin [MR22; McP21])

By using a certain presentation of $\mathcal{U}_\xi(\mathfrak{sl}_2)$, we can explicitly compute braiding matrices in terms of quantum dilogarithms.

- The (cyclic) quantum dilogarithm of Faddeev and Kashaev [FK94] is a matrix-valued function analogous to the dilogarithm function appearing in the computation of complex volume
- This computation should similarly be understood in terms of hyperbolic geometry
- This is a work in progress; a preliminary version is in my thesis [McP21]

Ideal triangulations and the braiding

- To describe hyperbolic structures on a link complement, use ideal triangulation [Thu80]
- To obtain these from link diagrams, use octahedral decomposition of Thurston [Thu99] and Kim, Kim, and Yoon [KKY18]
- Each crossing has four tetrahedra
- Our quantum braiding at a crossing factors into four quantum dilogarithms, one for each tetrahedron
- Suggests a close relationship (perhaps equivalence?) with invariants of Baseilhac and Benedetti [BB05]

Fixing the phase ambiguity

- Central characters χ of $\mathcal{U}_\xi(\mathfrak{sl}_2)$ parametrizing modules are closely related to shapes of hyperbolic ideal tetrahedron [McP22b]
- Phase ambiguity in $F_r(L, \rho, \omega)$ is related to picking rth roots of the shapes
- Analogous problem in computation of Chern-Simons invariant is solved by flattenings of Neumann [Neu04]
- I am currently working on using these to resolve the phase ambiguity

Going the other direction, we can apply ideas from geometric quantum field theory to hyperbolic geometry:

Theorem (Me, to appear)

For M a compact, oriented, closed 3-manifold and $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ there is a refined complex volume $\mathcal{V}(M, \rho) \in \mathbb{C}$ lifting the usual one:

$$\mathcal{V}(M, \rho) \equiv \mathrm{Vol}(M, \rho) + i\, \mathrm{CS}(M, \rho) \pmod{\pi^2 i \mathbb{Z}}$$

Recall that $\mathrm{CS}(M, \rho)$ is only defined $\mod \pi^2\mathbb{Z}$.

- $\mathcal{V}$ is also defined for manifolds with torus boundary given a choice of boundary conditions.
- It obeys gluing relations, so we can think of this as a geometric (classical) field theory

Proof idea

The proof comes from an analogy to the quantum invariant F_r.

- The description of the hyperbolic structure ρ in terms of $\mathcal{U}_\xi(\mathfrak{sl}_2)$ is also convenient for computing complex volume.
- Can use to understand $\pi^2 i$ ambiguity (analogous to phase ambiguity in F_r) and eliminate, then
- use techniques of Blanchet, Geer, Patureau-Mirand, and Reshetikhin [Bla+20] to prove this gives an invariant of (M, ρ).

Conclusion

- Motivated by volume conjecture and physics we want to upgrade TQFTs to include geometric data
- We call the values on links and manifold quantum holonomy invariants
- To define them, need to understand unusual representations of $\mathcal{U}_\xi(\mathfrak{sl}_2)$ at root of unity
- These are closely related to hyperbolic geometry and octahedral decompositions
- In the future, I hope these connections produce a better understanding of both quantum topology and of hyperbolic geometry/topology

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