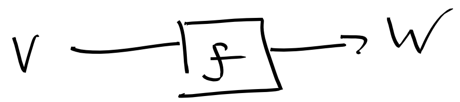
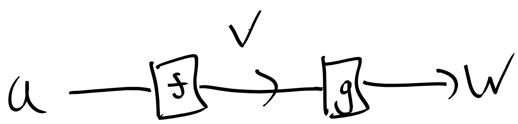
How to compute the colored Jones polynomial

String diagrams for linear maps: (U, V, W $\mathbb{C}$-vector spaces)

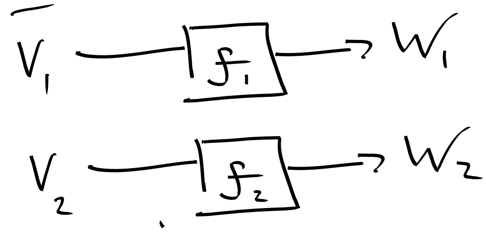


$f : V \to W$

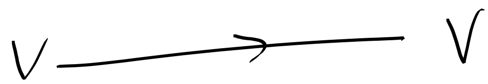


$g \circ f : U \to W$

← should be fg if function notation was correct



$f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$

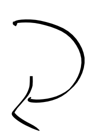


id_V

id_{V^*}

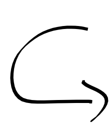
← nothing

$\mathbb{C}$



$ev : V \otimes V^* \to \mathbb{C}$

$ev(v \otimes f) = f(v)$

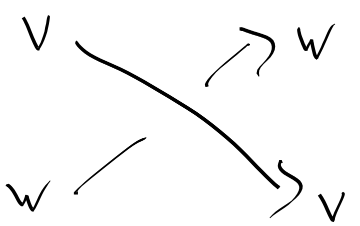


$coev : \mathbb{C} \to V^* \otimes V$

$coev(1) = \sum_i v_i \otimes v^i$

$v^i(v_j) = \delta_{ij}$

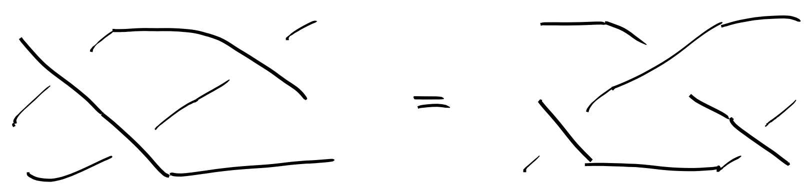
If V, W are modules for a <u>quantum group</u> $U_q(\mathfrak{g})$ (i.e. a q-analogue of a rep of Lie algebra $\mathfrak{g}$) then there is a <u>braiding</u>



$c_{V,W} : V \otimes W \to W \otimes V$

(more generally this is a "braided monoidal category". main examples come from $U_q(\mathfrak{g})$)

satisfying



$(c \otimes id)(id \otimes c)(c \otimes id) \qquad (id \otimes c)(c \otimes id)(id \otimes c)$

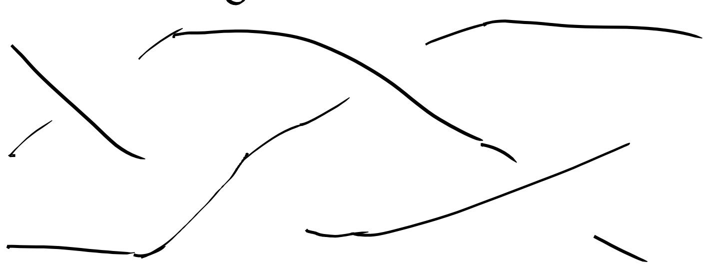
"braid relation" "Reidemeister III move"

"Yang-Baxter equation"

Comes from special element $R \in U_q \hat{\otimes} U_q$ (completed tensor product)

<u>universal R-matrix</u>.

Via our dictionary, a braid diagram



$\rightsquigarrow$ linear map $f : V^{\otimes 3} \to V^{\otimes 3}$

closure is like a trace, gives scalar

$\rightsquigarrow$ "quantum trace"

$tr_q f : \mathbb{C} \to \mathbb{C}$

$=$

scalar

$=$

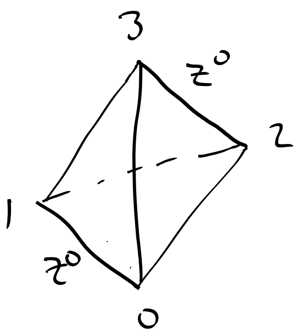
quantum invariant of our knot assoc. to K.

Computing hyperbolic structures and volumes of links

$K \subset S^3$ knot. Think of K lying "at ∞".

<u>Ideally</u> <u>triangulate</u> $S^3 \setminus K$: by tetrahedra w/ 0-skeletons missing, all lying on K.

To geometrize: Put vertices in Riemann sphere $\mathbb{C}P^1 = \partial \mathbb{H}^3$ ← hyperbolic space



geometry summarized by <u>shape parameter</u> $z = z^0 = \dfrac{(x_0 - x_3)(x_1 - x_2)}{(x_0 - x_2)(x_1 - x_3)}$

cross ratio is the function of 4 points invariant under fractional linear transfs $PSL_2(\mathbb{C}) = \mathrm{Isom}(\mathbb{C}P^1) = \mathrm{Isom}(\mathbb{H}^3)$

$\arg z^0$ = dihedral angle at edge z^0 = "complexified angle"

$|z^0|$ related to scaling

parameters $z^1 = \dfrac{1}{1-z^0}$, $z^2 = \dfrac{1}{1-z^1} = 1 - \dfrac{1}{z^0}$ assigned to other edges

To get consistent hyperbolic structure on $S^3 \setminus K$, shapes must satisfy

<u>Thurston's gluing equations</u> $\prod_i z_i^{k_i}$ for each edge of $S^3 \setminus K$.

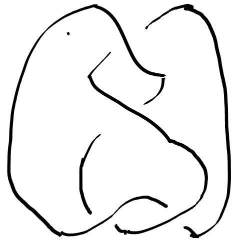
product over all tetrahedron edges meeting here

$k_i = 0, 1, 2$ depending on combinatorics

<u>Also</u>, can use shapes to understand behaviour on boundary torus of $S^3 \setminus \nu(K)$. Lots of important geometric info.

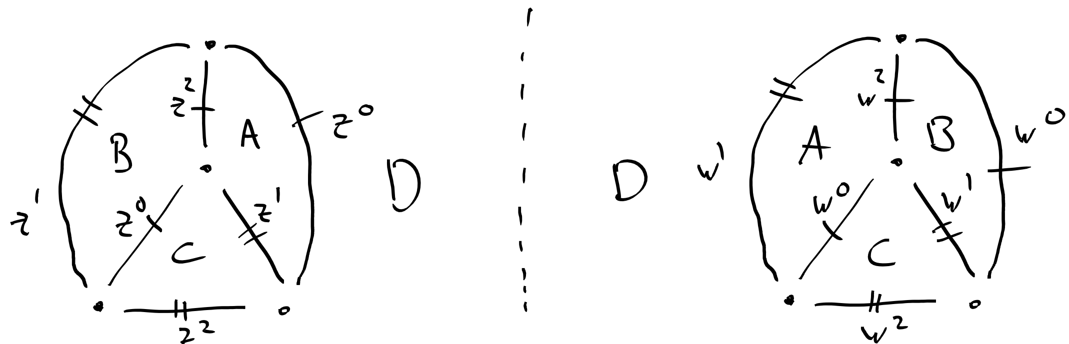
In particular, this method leads to hyperbolic structures on Dehn fillings (infinitely many of them!) of K.

Example: Figure-eight knot 4_1



Simplest hyperbolic knot: $S^3 \setminus 4_1$ admits a complete Riemannian metric of curvature -1 and finite volume.

Non-obvious fact: $S^3 \setminus 4_1$ admits an ideal triangulation with two tetrahedra $\leftarrow$ see Purcell, arXiv: 2002.12652 for details (also a great reference in general!)



glue faces, edges to get $S^3 \setminus$ figure-eight

edge $+$ gives $(z^0)^2 z^2 (w^0)^2 w^2 = 1 \implies z(z-1)w(w-1) = 1 \implies$

edge $\#$ gives $(z^1)^2 z^2 (w^1)^2 w^2 = 1$ $\quad z = \dfrac{1 \pm \sqrt{1+4/w(w-1)}}{2}$

implies this one too

Particularly interested in $z = w = \exp(\pi i/3) = \frac{1+\sqrt{-3}}{2}$. Gives complete structure. Nearby solutions give structure on Dehn filled manifolds. (To figure this out: look at boundary tori.)

"Bloch-Wigner function" $D(z)$

Volume of one tetrahedron w/ param z is $D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$

"dilogarithm" $\mathrm{Li}_2(z) := \int_0^z -\frac{\log(1-t)}{t}\,dt = \sum_{n=1}^{\infty} z^n/n^2$

We get $\mathrm{Vol}(S^3 \setminus 4_1) = 2D(e^{\pi i/3}) \approx 2.02988\ldots$

How to do this in general? Via computers! SnapPy

Straight from knot diagrams: octahedral decompositions arXiv: 2203.06042